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一类非线性变系数 Burgers 方程初边值问题的差分方法

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摘 要: 针对一维非线性变系数 Burgers 方程的初边值问题进行差分方法研究。首先, 利用差分方法, 建立二层非线性差分格式; 然后证明该差分格式在适当的条件下具有稳定性、收敛性; 最后, 通过具体数值算例, 验证了本文构造的差分方法的有效性。算例结果表明该差分格式在空间和时间上都具有 2 阶精度, 展示了其在计算精度、稳定性等方面的优势。

关键词: 计算数学; 非线性变系数 Burgers 方程; 二层格式; 非线性差分格式; 稳定性; 收敛性

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The Difference Method for the Initial Boundary Value Problem of a Nonlinear Variable Coefficient Burgers Equations

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Abstract: An inquiry has been made into the difference method for the initial boundary value problem of one-dimensional nonlinear variable coefficient Burgers equation. Firstly, by using the difference method, a two-layer nonlinear difference scheme has been established, followed by a proof of stability and convergence under appropriate conditions to be possessed by the proposed difference scheme. Finally, the effectiveness of the differential method constructed in this current research can be verified through specific numerical examples. The results indicate that the differential scheme is characterized with second-order accuracy in both space and time, thus demonstrating its advantages in computational accuracy, stability, and other aspects.

Keywords: computational mathematics; nonlinear variable coefficient Burgers equation; two-level difference scheme; nonlinear difference scheme; stability; convergence

1 研究背景

随着社会的进步, 非线性微分方程在包括化学、生物学、电子技术等自然科学和技术科学领域中的作

用日益显著。这些领域中都涉及大量的非线性微分方程, 因此, 探索非线性微分方程的解析解与数值解成为一个至关重要的研究课题。而 Burgers 方程是一类重要的非线性偏微分方程, 它是描述许多物理现象

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的模型方程,如流体动力学和湍流研究、气体动力学、交通流动力学等。在流体动力学和湍流研究中,Burgers 方程被用来研究流体中湍流现象的某些方面,尤其是在理解如何从不规则的初始条件发展出复杂时空结构的过程中;在气体动力学中,Burgers 方程可以用来描述某些一维流动问题,例如激波的形成和传播;它也可以用来研究气体流动中的非线性波现象,如非线性声波;在交通工程中,Burgers 方程可以用来模拟高速公路上的交通流量,描述车辆密度和速度的变化。

针对一维非线性 Burgers 方程初边值问题的求解具有一定的挑战性,因为它通常涉及非线性项,这使得直接求解变得复杂。因此,为了有效地解决这类问题,研究者们发展了多种数值方法,其中差分方法是一种常见且重要的方法。忻孝康等^[1]探讨了一维 Burgers 方程的不同差分格式。文献[2-4]发展了一维 Burgers 方程的多种分组显式差分格式。文献[5]对一维 Burgers 方程的有限元方法进行了探讨。文献[6]对一维 Burgers 方程的有限元逼近方法进行了最优误差估计和超收敛性质的分析。文献[7]采用混合有限元方法对一维 Burgers 方程进行了数值研究。文献[8]提出了一维 Burgers 方程的谱方法。文献[9]对二维 Burgers 方程的数值求解进行了研究。张伟等^[10]使用格子 Boltzmann 方法求解了一维黏性 Burgers 方程。

文献[11]探讨了如何求超广义黏性 Burgers 方程的精确解,并对其在数值计算过程中产生的误差进行了评估。文献[12]研究了超广义黏性 Burgers 方程的三层保守差分格式的逐点误差估计。文献[13]研究了四阶分数波动方程的快速紧差分格式。文献[14]研究了 Burgers 方程的紧致差分方法。文献[15]研究了如何利用基于紧致差分格式的 Richardson 外推方法来解决椭圆型偏微分方程。文献[16]针对一类非线性 Burgers 方程,提出并分析了一种预测校正紧差分格式。文献[17]研究了带界面椭圆最优控制问题的自适应有限元方法。文献[18]采用函数方法讨论了变时滞反馈控制下的 HNS-DDEs 的指数稳定性。文献[19]研究了广义 Burgers 方程的差分方法。

本文主要研究变系数 Burgers 方程高阶精度数值格式的构造,研究的模型如下:

$$\begin{aligned} u_t + uu_x &= v(x,t)u_{xx} + p(x,t)u_x, \\ 0 < x < L_1, \quad 0 < t \leq T_1, \end{aligned} \quad (1)$$

式中 $v(x,t)$ 为扩散系数,且 $v(x,t) \geq 0$ 。

模型的初始条件和边界条件如下:

$$u(x,0) = \varphi(x), \quad 0 < x < L_1; \quad (2)$$

$$u(0,t) = 0, \quad u(L_1,t) = 0, \quad 0 \leq t \leq T_1. \quad (3)$$

2 预备知识

为了用差分格式求解模型(1),将区间 $[0, L_1]$ 作 n_1 等分,记 $h=L_1/n_1$, $x_i=ih$, $0 \leq i \leq n_1$; 将 $[0, T_1]$ 作 n_2 等分,记 $\tau=T_1/n_2$, $t_k=k\tau$, $0 \leq k \leq n_2$;

$$\Omega_h = \{x_i \mid 0 \leq i \leq n_1\};$$

$$\Omega_\tau = \{t_k \mid 0 \leq k \leq n_2\};$$

$$\Omega_{h\tau} = \Omega_h \times \Omega_\tau;$$

$$U_i^k = u(x_i, t_k), \quad 0 \leq i \leq n_1, \quad 0 \leq k \leq n_2;$$

$$v_i^{k+1/2} = v(x_i, t_{k+1/2}), \quad 0 \leq i \leq n_1, \quad 0 \leq k \leq n_2,$$

$$p_i^{k+1/2} = p(x_i, t_{k+1/2}), \quad 0 \leq i \leq n_1, \quad 0 \leq k \leq n_2.$$

设 $u \in U_h$, 引进如下记号:

$$x_{i+1/2} = (x_i + x_{i+1})/2, \quad t_{k+1/2} = (t_k + t_{k+1})/2.$$

记

$$U_h = \{u \mid u = (u_0, u_1, \dots, u_m) \text{ 为 } \Omega_h \text{ 上的网格函数}\},$$

$$\dot{U}_h = \{u \mid u \in U_h, u_0 = u_m = 0\},$$

$$\delta_x u_{i+1/2} = (u_{i+1} - u_i)/h, \quad \delta_x u_{i-1/2} = (u_i - u_{i-1})/h,$$

$$\delta_x v_{i+1/2} = (v_{i+1} - v_i)/h, \quad \delta_x v_{i-1/2} = (v_i - v_{i-1})/h,$$

$$\delta_x p_{i+1/2} = (p_{i+1} - p_i)/h, \quad \Delta_x u_i = (u_{i+1} - u_{i-1})/2h,$$

$$\delta_x^2 u_i = (u_{i-1} - 2u_i + u_{i+1})/h^2, \quad U_i^{k+1/2} = (U_i^k + U_i^{k+1})/2.$$

易知

$$\delta_x^2 u_i = (\delta_x u_{i+1/2} - \delta_x u_{i-1/2})/h, \quad \Delta_x u_i = (\delta_x u_{i-1/2} + \delta_x u_{i+1/2})/2.$$

设 $u, v \in U_h$, 引进内积、范数, 则有:

$$(u, v) = h \left(u_0 v_0 / 2 + \sum_{i=1}^{m-1} u_i v_i + u_m v_m / 2 \right), \quad \|u\| = \sqrt{(u, u)}.$$

引理 1^[20] 设 c, h 为给定的常数, 且 $h > 0$,

I) 如果函数

$$g \in C^2[c-h, c+h],$$

则有

$$\begin{aligned} g(c) &= [g(c-h) + g(c+h)]/2 - h^2 g''(\xi_0)/2, \\ c-h &< \xi_0 < c+h. \end{aligned}$$

II) 如果函数

$$g \in C^3[c-h, c+h],$$

则有

$$\begin{aligned} g'(c) &= [g(c+h) - g(c-h)]/2h - h^2 g'''(\xi_1)/6, \\ c-h &< \xi_1 < c+h. \end{aligned}$$

引理 2^[20] (Gronwall 不等式) 设 a 和 b 是两个

非负常数, $\{F^k\}_{k=0}^\infty$ 是一个不是负数的序列, 而且

$$F^{k+1} \leq (1 + a\tau)F^k + \tau b, \quad k = 0, 1, 2, \dots,$$

那么

$$F^k \leq e^{ak\tau} \left(F^0 + \frac{b}{a} \right), \quad k = 1, 2, 3, \dots$$

记

$$c_1 = \max \left\{ \max_{0 \leq x \leq L_1, 0 \leq t \leq T_1} \left| \frac{\partial^2 v(x, t)}{\partial x^2} \right|, \max_{0 \leq x \leq L_1, 0 \leq t \leq T_1} \left| \frac{\partial p(x, t)}{\partial x} \right| \right\}.$$

引理 3 对于 $u \in \bar{U}_h$, 有以下不等式成立:

$$\left(v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2} \right) \leq -h \sum_{i=1}^{n_1-1} v_i^{k+1/2} \left(\Delta_x u_i^{k+1/2} \right)^2 + c_1 \|u^{k+1/2}\|^2 / 2.$$

证明

$$\begin{aligned} \left(v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2} \right) &= h \sum_{i=1}^{n_1-1} v_i^{k+1/2} \left(\delta_x^2 u_i^{k+1/2} \right) u_i^{k+1/2} = \\ &= h \sum_{i=1}^{n_1-1} v_i^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} - \delta_x u_{i-1/2}^{k+1/2} \right) u_i^{k+1/2} / h = \\ &= \sum_{i=1}^{n_1-1} v_i^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} - \delta_x u_{i-1/2}^{k+1/2} \right) u_i^{k+1/2} = \\ &= \sum_{i=1}^{n_1-1} v_i^{k+1/2} u_i^{k+1/2} \delta_x u_{i+1/2}^{k+1/2} - \sum_{i=1}^{n_1-1} v_i^{k+1/2} u_i^{k+1/2} \delta_x u_{i-1/2}^{k+1/2} = \\ &= \sum_{i=1}^{n_1-1} v_i^{k+1/2} u_i^{k+1/2} \delta_x u_{i+1/2}^{k+1/2} - \sum_{i=0}^{n_1-2} v_{i+1}^{k+1/2} u_{i+1}^{k+1/2} \delta_x u_{i+1/2}^{k+1/2} = \\ &= \sum_{i=0}^{n_1-1} \left(v_i^{k+1/2} u_i^{k+1/2} - v_{i+1}^{k+1/2} u_{i+1}^{k+1/2} \right) \delta_x u_{i+1/2}^{k+1/2} - \\ &= v_0^{k+1/2} u_0^{k+1/2} \delta_x u_{1/2}^{k+1/2} - v_{n_1}^{k+1/2} u_{n_1}^{k+1/2} \delta_x u_{n_1-1/2}^{k+1/2} = \\ &= \sum_{i=0}^{n_1-1} \left(v_i^{k+1/2} u_i^{k+1/2} - v_{i+1}^{k+1/2} u_{i+1}^{k+1/2} \right) \delta_x u_{i+1/2}^{k+1/2}, \end{aligned}$$

利用

$$u_i^{k+1/2} = u_{i+1}^{k+1/2} - h \delta_x u_{i+1/2}^{k+1/2},$$

则上式可进一步化为

$$\begin{aligned} \sum_{i=0}^{n_1-1} \left[v_i^{k+1/2} \left(u_{i+1}^{k+1/2} - h \delta_x u_{i+1/2}^{k+1/2} \right) - v_{i+1}^{k+1/2} u_{i+1}^{k+1/2} \right] \delta_x u_{i+1/2}^{k+1/2} = \\ \sum_{i=0}^{n_1-1} \left[-h v_i^{k+1/2} \delta_x u_{i+1/2}^{k+1/2} + u_{i+1}^{k+1/2} \left(v_i^{k+1/2} - v_{i+1}^{k+1/2} \right) \right] \delta_x u_{i+1/2}^{k+1/2} = \\ h \sum_{i=0}^{n_1-1} - \left(v_i^{k+1/2} \delta_x u_{i+1/2}^{k+1/2} + u_{i+1}^{k+1/2} \delta_x v_{i+1/2}^{k+1/2} \right) \delta_x u_{i+1/2}^{k+1/2} = \\ h \sum_{i=0}^{n_1-1} - v_i^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} \right)^2 - h \sum_{i=0}^{n_1-1} \delta_x v_{i+1/2}^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} \right) u_{i+1}^{k+1/2}, \end{aligned}$$

又因为

$$\begin{aligned} \left(v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2} \right) &= h \sum_{i=1}^{n_1-1} v_i^{k+1/2} \left(\delta_x^2 u_i^{k+1/2} \right) u_i^{k+1/2} = \\ &= h \sum_{i=1}^{n_1-1} v_i^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} - \delta_x u_{i-1/2}^{k+1/2} \right) u_i^{k+1/2} / h = \\ &= \sum_{i=1}^{n_1-1} v_i^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} - \delta_x u_{i-1/2}^{k+1/2} \right) u_i^{k+1/2} = \\ &= \sum_{i=1}^{n_1-1} v_i^{k+1/2} u_i^{k+1/2} \delta_x u_{i+1/2}^{k+1/2} - \sum_{i=1}^{n_1-1} v_i^{k+1/2} u_i^{k+1/2} \delta_x u_{i-1/2}^{k+1/2} = \\ &= \sum_{i=2}^{n_1} v_{i-1}^{k+1/2} u_{i-1}^{k+1/2} \delta_x u_{i-1/2}^{k+1/2} - \sum_{i=1}^{n_1-1} v_i^{k+1/2} u_i^{k+1/2} \delta_x u_{i-1/2}^{k+1/2} = \\ &= \sum_{i=1}^{n_1} \left(v_{i-1}^{k+1/2} u_{i-1}^{k+1/2} - v_i^{k+1/2} u_i^{k+1/2} \right) \delta_x u_{i-1/2}^{k+1/2} - \\ &= v_0^{k+1/2} u_0^{k+1/2} \delta_x u_{1/2}^{k+1/2} - v_{n_1}^{k+1/2} u_{n_1}^{k+1/2} \delta_x u_{n_1-1/2}^{k+1/2} = \\ &= \sum_{i=1}^{n_1} \left(v_{i-1}^{k+1/2} u_{i-1}^{k+1/2} - v_i^{k+1/2} u_i^{k+1/2} \right) \delta_x u_{i-1/2}^{k+1/2}, \end{aligned}$$

利用

$$u_i^{k+1/2} = u_{i-1}^{k+1/2} + h \delta_x u_{i-1/2}^{k+1/2},$$

则上式可进一步化为

$$\begin{aligned} \sum_{i=1}^{n_1} \left[v_{i-1}^{k+1/2} u_{i-1}^{k+1/2} - v_i^{k+1/2} \left(u_{i-1}^{k+1/2} + h \delta_x u_{i-1/2}^{k+1/2} \right) \right] \delta_x u_{i-1/2}^{k+1/2} = \\ \sum_{i=1}^{n_1} \left[-h \delta_x u_{i-1/2}^{k+1/2} + u_{i-1}^{k+1/2} \left(v_{i-1}^{k+1/2} - v_i^{k+1/2} \right) \right] \delta_x u_{i-1/2}^{k+1/2} = \\ h \sum_{i=1}^{n_1} - \left(v_i^{k+1/2} \delta_x u_{i-1/2}^{k+1/2} + u_{i-1}^{k+1/2} \delta_x v_{i-1/2}^{k+1/2} \right) \delta_x u_{i-1/2}^{k+1/2} = \\ h \sum_{i=1}^{n_1} - v_i^{k+1/2} \left(\delta_x u_{i-1/2}^{k+1/2} \right)^2 - h \sum_{i=1}^{n_1} \delta_x v_{i-1/2}^{k+1/2} \left(\delta_x u_{i-1/2}^{k+1/2} \right) u_{i-1}^{k+1/2} = \\ h \sum_{i=1}^{n_1} - v_i^{k+1/2} \left(\delta_x u_{i-1/2}^{k+1/2} \right)^2 - h \sum_{i=0}^{n_1-1} \delta_x v_{i+1/2}^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} \right) u_i^{k+1/2}, \end{aligned}$$

联系上面两式, 可得:

$$\begin{aligned} \left(v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2} \right) &= \\ &= h \left[- \sum_{i=0}^{n_1-1} v_i^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} \right)^2 - \sum_{i=1}^{n_1} v_i^{k+1/2} \left(\delta_x u_{i-1/2}^{k+1/2} \right)^2 \right] - \\ &= h \sum_{i=0}^{n_1-1} \delta_x v_{i+1/2}^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} \right) \left(u_{i+1}^{k+1/2} + u_i^{k+1/2} \right) / 2 = \\ &= h \sum_{i=0}^{n_1-1} - v_i^{k+1/2} \left[\left(\delta_x u_{i+1/2}^{k+1/2} \right)^2 + \left(\delta_x u_{i-1/2}^{k+1/2} \right)^2 \right] - \\ &= h v_{n_1}^{k+1/2} \left(\delta_x u_{n_1-1/2}^{k+1/2} \right)^2 - h \sum_{i=0}^{n_1-1} \delta_x v_{i+1/2}^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} \right) \cdot \\ &= \left(u_{i+1}^{k+1/2} + u_i^{k+1/2} \right) / 2 \leq \\ &= -h \sum_{i=1}^{n_1-1} v_i^{k+1/2} \left(\left(\delta_x u_{i+1/2}^{k+1/2} \right)^2 + \left(\delta_x u_{i-1/2}^{k+1/2} \right)^2 \right) / 2 - \\ &= h \sum_{i=0}^{n_1-1} \delta_x v_{i+1/2}^{k+1/2} \left(\delta_x u_{i+1/2}^{k+1/2} \right) \left(u_{i+1}^{k+1/2} + u_i^{k+1/2} \right) / 2 = \\ &= h \sum_{i=1}^{n_1-1} - v_i^{k+1/2} \left(\left(\delta_x u_{i+1/2}^{k+1/2} \right)^2 + \left(\delta_x u_{i-1/2}^{k+1/2} \right)^2 \right) / 2 - \end{aligned}$$

$$\begin{aligned}
& h \sum_{i=0}^{n_1-1} \delta_x v_{i+1/2}^{k+1/2} (u_{i+1}^{k+1/2} - u_i^{k+1/2}) (u_{i+1}^{k+1/2} + u_i^{k+1/2}) / 2 / h = \\
& h \sum_{i=1}^{n_1-1} -v_i^{k+1/2} \left((\delta_x u_{i+1/2}^{k+1/2})^2 + (\delta_x u_{i-1/2}^{k+1/2})^2 \right) / 2 - \\
& \sum_{i=0}^{n_1-1} \delta_x v_{i+1/2}^{k+1/2} \left((u_{i+1}^{k+1/2})^2 - (u_i^{k+1/2})^2 \right) / 2 \leq \\
& h \sum_{i=1}^{n_1-1} -v_i^{k+1/2} (\Delta_x u_i^{k+1/2})^2 - \\
& \sum_{i=1}^{n_1-1} (\delta_x v_{i-1/2}^{k+1/2} - \delta_x v_{i+1/2}^{k+1/2}) (u_i^{k+1/2})^2 / 2,
\end{aligned}$$

因为

$$|\delta_x v_{i-1/2}^{k+1/2} - \delta_x v_{i+1/2}^{k+1/2}| \leq c_1 h,$$

所以

$$\begin{aligned}
(v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2}) & \leq h \sum_{i=1}^{n_1-1} -v_i^{k+1/2} (\Delta_x u_i^{k+1/2})^2 - \\
& \sum_{i=1}^{n_1-1} (\delta_x v_{i-1/2}^{k+1/2} - \delta_x v_{i+1/2}^{k+1/2}) (u_i^{k+1/2})^2 / 2 \leq \\
& h \sum_{i=1}^{n_1-1} -v_i^{k+1/2} (\Delta_x u_i^{k+1/2})^2 + c_1 \|u^{k+1/2}\|^2 / 2.
\end{aligned}$$

引理 4 对于 $u \in \dot{U}_h$, 有

$$(p^{k+1/2} \Delta_x u^{k+1/2}, u^{k+1/2}) \leq c_1 \|u^{k+1/2}\|^2 / 2.$$

证明

$$\begin{aligned}
(p^{k+1/2} \Delta_x u^{k+1/2}, u^{k+1/2}) & = h \sum_{i=1}^{n_1-1} p_i^{k+1/2} (\Delta_x u_i^{k+1/2}) u_i^{k+1/2} = \\
& h \sum_{i=1}^{n_1-1} p_i^{k+1/2} (u_{i+1}^{k+1/2} - u_{i-1}^{k+1/2}) u_i^{k+1/2} / 2 / h = \\
& \sum_{i=1}^{n_1-1} p_i^{k+1/2} (u_{i+1}^{k+1/2} - u_{i-1}^{k+1/2}) u_i^{k+1/2} / 2 = \\
& \sum_{i=1}^{n_1-1} p_i^{k+1/2} u_{i+1}^{k+1/2} u_i^{k+1/2} / 2 - \sum_{i=1}^{n_1-1} p_i^{k+1/2} u_i^{k+1/2} u_{i-1}^{k+1/2} / 2 = \\
& \sum_{i=1}^{n_1-1} p_i^{k+1/2} u_{i+1}^{k+1/2} u_i^{k+1/2} / 2 - \sum_{i=0}^{n_1-2} p_{i+1}^{k+1/2} u_{i+1}^{k+1/2} u_i^{k+1/2} / 2 = \\
& \sum_{i=1}^{n_1-1} (p_i^{k+1/2} - p_{i+1}^{k+1/2}) u_{i+1}^{k+1/2} u_i^{k+1/2} / 2 - \\
& p_0^{k+1/2} u_1^{k+1/2} u_0^{k+1/2} / 2 + p_{n_1}^{k+1/2} u_{n_1}^{k+1/2} u_{n_1-1}^{k+1/2} / 2 = \\
& \sum_{i=1}^{n_1-1} (p_i^{k+1/2} - p_{i+1}^{k+1/2}) u_{i+1}^{k+1/2} u_i^{k+1/2} / 2 = \\
& \sum_{i=1}^{n_1-1} -h (\delta_x p_{i+1/2}^{k+1/2}) u_{i+1}^{k+1/2} u_i^{k+1/2} / 2 \leq \\
& h \sum_{i=1}^{n_1-1} c_1 \left((u_{i+1}^{k+1/2})^2 + (u_i^{k+1/2})^2 \right) / 4 \leq \frac{c_1}{2} \|u^{k+1/2}\|^2.
\end{aligned}$$

引理 5 对于 $u \in \dot{U}_h$, 则有下式成立

$$(p^{k+1/2} \Delta_x u^{k+1/2}, u^{k+1/2}) + (v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2}) \leq c_1 \|u^{k+1/2}\|^2.$$

证明 由引理 3 和引理 4 得

$$\begin{aligned}
& (p^{k+1/2} \Delta_x u^{k+1/2}, u^{k+1/2}) + (v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2}) \leq \\
& c_1 \|u^{k+1/2}\|^2 / 2 - h \sum_{i=1}^{n_1-1} v_i^{k+1/2} (\Delta_x u_i^{k+1/2})^2 + \\
& c_1 \|u^{k+1/2}\|^2 / 2 \leq c_1 \|u^{k+1/2}\|^2.
\end{aligned}$$

引理 6^[20] 假设 $v \in U_h$, $w \in \dot{U}_h$, 定义:

$$\begin{aligned}
\psi(v^{k+1/2}, w^{k+1/2})_i & = [v_i^{k+1/2} \Delta_x w_i^{k+1/2} + \Delta_x (v^{k+1/2} w^{k+1/2})_i] / 3, \\
1 & \leq i \leq n_1 - 1,
\end{aligned}$$

则有

$$(\psi(v, w), w) = 0.$$

3 差分格式建立

在 $(x_i, t_{k+1/2})$ 考虑方程 (1), 有

$$\begin{aligned}
& u_t(x_i, t_{k+1/2}) + u(x_i, t_{k+1/2}) u_x(x_i, t_{k+1/2}) = \\
& v(x_i, t_{k+1/2}) u_{xx}(x_i, t_{k+1/2}) + p(x_i, t_{k+1/2}) u_x(x_i, t_{k+1/2}), \\
& 1 \leq i \leq n_1 - 1, 0 \leq k \leq n_2 - 1,
\end{aligned} \quad (4)$$

由引理 1 得

$$u_t(x_i, t_{k+1/2}) = \delta_t U_i^{k+1/2} + O(\tau^2), \quad (5)$$

因为

$$\begin{aligned}
& u(x_i, t_{k+1/2}) = \\
& [u(x_{i-1}, t_{k+1/2}) + u(x_i, t_{k+1/2}) + u(x_{i+1}, t_{k+1/2})] / 3 + \\
& O(h^2) = (U_{i-1}^{k+1/2} + U_i^{k+1/2} + U_{i+1}^{k+1/2}) / 3 + O(\tau^2 + h^2), \\
& u_x(x_i, t_{k+1/2}) = [u_x(x_i, t_k) + u_x(x_i, t_{k+1})] / 2 + O(\tau^2) = \\
& (\Delta_x U_i^k + \Delta_x U_i^{k+1}) / 2 + O(\tau^2 + h^2) = \\
& \Delta_x U_i^{k+1/2} + O(\tau^2 + h^2),
\end{aligned}$$

所以

$$\begin{aligned}
& u(x_i, t_{k+1/2}) u_x(x_i, t_{k+1/2}) = \\
& [(U_{i-1}^{k+1/2} + U_i^{k+1/2} + U_{i+1}^{k+1/2}) / 3 + O(\tau^2 + h^2)] \cdot \\
& [\Delta_x U_i^{k+1/2} + O(\tau^2 + h^2)] = (U_{i-1}^{k+1/2} + U_i^{k+1/2} + U_{i+1}^{k+1/2}) \cdot \\
& \Delta_x U_i^{k+1/2} / 3 + O(\tau^2 + h^2) = \\
& [U_i^{k+1/2} \Delta_x U_i^{k+1/2} + (U_{i+1}^{k+1/2} + U_{i-1}^{k+1/2}) \Delta_x U_i^{k+1/2}] / 3 + \\
& O(\tau^2 + h^2) = \\
& [U_i^{k+1/2} \Delta_x U_i^{k+1/2} + \Delta_x (U_i^{k+1/2} U_i^{k+1/2})] / 3 + \\
& O(\tau^2 + h^2) = \psi(U^{k+1/2}, U^{k+1/2})_i + O(\tau^2 + h^2).
\end{aligned} \quad (6)$$

$$\begin{aligned}
& v(x_i, t_{k+1/2}) u_{xx}(x_i, t_{k+1/2}) = \\
& v_i^{k+1/2} [u_{xx}(x_i, t_k) + u_{xx}(x_i, t_{k+1})] / 2 + O(\tau^2) = \\
& v_i^{k+1/2} (\delta_x^2 U_i^k + \delta_x^2 U_i^{k+1}) / 2 + O(\tau^2 + h^2) = \\
& v_i^{k+1/2} \delta_x^2 U_i^{k+1/2} + O(\tau^2 + h^2).
\end{aligned} \quad (7)$$

$$\begin{aligned}
p(x_i, t_{k+1/2})u_x(x_i, t_{k+1/2}) &= \\
p_i^{k+1/2} [u_x(x_i, t_k) + u_x(x_i, t_{k+1})] / 2 + O(\tau^2) &= \\
p_i^{k+1/2} (\Delta_x U_i^k + \Delta_x U_i^{k+1}) / 2 + O(\tau^2 + h^2) &= \\
p_i^{k+1/2} \Delta_x U_i^{k+1/2} + O(\tau^2 + h^2). & \quad (8)
\end{aligned}$$

将式(5)~(8)代入式(4), 得到

$$\begin{aligned}
\delta_i U_i^{k+1/2} + \psi(U^{k+1/2}, U^{k+1/2})_i &= \\
v_i^{k+1/2} \delta_x^2 U_i^{k+1/2} + p_i^{k+1/2} \Delta_x U_i^{k+1/2} + (R_1)_i^k, & \\
1 \leq i \leq n_1 - 1, 0 \leq k \leq n_2 - 1, & \quad (9)
\end{aligned}$$

存在常数 c_2 使得

$$|(R_1)_i^k| \leq c_2(\tau^2 + h^2), 1 \leq i \leq n_1 - 1, 0 \leq k \leq n_2 - 1. \quad (10)$$

注意到初边值条件(2)和(3), 有

$$\begin{cases} U_i^0 = \varphi(x_i), 1 \leq i \leq n_1 - 1; \\ U_0^k = 0, U_m^k = 0, 0 \leq k \leq n_2. \end{cases} \quad (11)$$

略去小量项 $(R_1)_i^k$, 用 u_i^k 代替 U_i^k , 对式(1)建立如下差分格式:

$$\begin{aligned}
\delta_i u_i^{k+1/2} + \psi(u^{k+1/2}, u^{k+1/2})_i &= \\
v_i^{k+1/2} \delta_x^2 u_i^{k+1/2} + p_i^{k+1/2} \Delta_x u_i^{k+1/2}, & \\
1 \leq i \leq n_1 - 1, 0 \leq k \leq n_2 - 1; & \quad (12)
\end{aligned}$$

$$u_i^0 = \varphi(x_i), 1 \leq i \leq n_1 - 1; \quad (13)$$

$$u_0^k = 0, u_m^k = 0, 0 \leq k \leq n_2. \quad (14)$$

4 差分格式理论分析

定理 1 (稳定性)

设 $\{u_i^k | 0 \leq i \leq n_1, 0 \leq k \leq n_2\}$ 为差分格式(12)~(13)的解, 则当 $\tau < 1/c_1$ 时, 有

$$\|u^k\| \leq \exp(2c_1 k \tau) \|u^0\|. \quad (15)$$

证明 将式(12)与 $u^{k+1/2}$ 做内积, 得到

$$\begin{aligned}
(\delta_i u_i^{k+1/2}, u^{k+1/2}) + (\psi(u^{k+1/2}, u^{k+1/2}), u^{k+1/2}) &= \\
(v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2}) + (p^{k+1/2} \Delta_x u^{k+1/2}, u^{k+1/2}), & \\
0 \leq k \leq n_2 - 1. &
\end{aligned}$$

因为

$$(\delta_i u_i^{k+1/2}, u^{k+1/2}) = (\|u^{k+1}\|^2 - \|u^k\|^2) / 2\tau,$$

注意到式(14), 由引理6得

$$(\psi(u^{k+1/2}, u^{k+1/2}), u^{k+1/2}) = 0.$$

利用上面两式、引理5和Cauchy-Schwarz不等式可得

$$\begin{aligned}
(\|u^{k+1}\|^2 - \|u^k\|^2) / 2\tau &= (v^{k+1/2} \delta_x^2 u^{k+1/2}, u^{k+1/2}) + \\
(p^{k+1/2} \Delta_x u^{k+1/2}, u^{k+1/2}) &\leq c_1 \|u^{k+1/2}\|^2 \leq \\
c_1 (\|u^{k+1} + u^k\| / 2)^2 &\leq c_1 ((\|u^{k+1}\| + \|u^k\|) / 2)^2, \\
0 \leq k \leq n_2 - 1. &
\end{aligned}$$

即

$$(\|u^{k+1}\|^2 - \|u^k\|^2) / 2\tau \leq c_1 ((\|u^{k+1}\| + \|u^k\|) / 2)^2,$$

于是,

$$(\|u^{k+1}\| - \|u^k\|) / \tau \leq c_1 (\|u^{k+1}\| + \|u^k\|) / 2, 0 \leq k \leq n_2 - 1.$$

也就是说,

$$(1 - c_1 \tau / 2) \|u^{k+1}\| \leq (1 + c_1 \tau / 2) \|u^k\|, 0 \leq k \leq n_2 - 1.$$

当 $\tau < 1/c_1$, 由此得出,

$$\|u^{k+1}\| \leq (1 + 2c_1 \tau) \|u^k\|, 0 \leq k \leq n_2 - 1.$$

利用引理2, 定理1得证。

定理 2 (收敛性)

设 $\{U_i^k | 0 \leq i \leq n_1, 0 \leq k \leq n_2\}$ 为问题的解,

$\{u_i^k | 0 \leq i \leq n_1, 0 \leq k \leq n_2\}$ 为差分格式(12)~(14)

的解。记 $e_i^k = U_i^k - u_i^k$, $0 \leq i \leq n_1$, $0 \leq k \leq n_2$, 则存在常数 c_3 使得

$$\|e^k\| \leq c_3(\tau^2 + h^2), 0 \leq k \leq n_2.$$

证明 把式(9)和(11)分别与式(12)~(14)做减法, 得到一个误差方程组

$$\begin{aligned}
\delta_i e_i^{k+1/2} + \psi(U^{k+1/2}, U^{k+1/2})_i - \psi(u^{k+1/2}, u^{k+1/2})_i &= \\
v_i^{k+1/2} \delta_x^2 e_i^{k+1/2} + p_i^{k+1/2} \Delta_x e_i^{k+1/2} + (R_1)_i^k, & \\
1 \leq i \leq n_1 - 1, 0 \leq k \leq n_2 - 1. & \quad (16)
\end{aligned}$$

$$e_i^k = 0, 1 \leq i \leq n_1 - 1. \quad (17)$$

$$e_i^k = 0, e_m^k = 0, 0 \leq k \leq n_2. \quad (18)$$

把 $e^{k+1/2}$ 与式(16)做内积, 得到,

$$\begin{aligned}
(\delta_i e_i^{k+1/2}, e^{k+1/2}) + (\psi(U^{k+1/2}, U^{k+1/2}) - \psi(u^{k+1/2}, u^{k+1/2}), e^{k+1/2}) &= \\
(v^{k+1/2} \delta_x^2 e^{k+1/2}, e^{k+1/2}) + (p^{k+1/2} \Delta_x e^{k+1/2}, e^{k+1/2}) &+ \\
((R_1)_i^k, e^{k+1/2}), & \\
0 \leq k \leq n_2 - 1. & \quad (19)
\end{aligned}$$

因为

$$(\delta_i e_i^{k+1/2}, e^{k+1/2}) = (\|e^{k+1}\|^2 - \|e^k\|^2) / 2\tau, \quad (20)$$

则式 (19) 可变形为

$$\begin{aligned} & (\|e^{k+1}\|^2 - \|e^k\|^2) / 2\tau = (\psi(U^{k+1/2}, U^{k+1/2}) - \\ & \psi(u^{k+1/2}, u^{k+1/2}), e^{k+1/2}) + (v^{k+1/2} \delta_x^2 e^{k+1/2}, e^{k+1/2}) + \\ & (p^{k+1/2} \Delta_x e^{k+1/2}, e^{k+1/2}) + ((R_1)^k, e^{k+1/2}), \\ & 0 \leq k \leq n_2 - 1. \end{aligned} \quad (21)$$

注意到

$$\begin{aligned} & \psi(U^{k+1/2}, U^{k+1/2}) - \psi(u^{k+1/2}, u^{k+1/2}) = \\ & \psi(U^{k+1/2}, U^{k+1/2}) - \\ & \psi(U^{k+1/2} - u^{k+1/2}, U^{k+1/2} - u^{k+1/2}) = \\ & \psi(e^{k+1/2}, U^{k+1/2}) + \psi(U^{k+1/2}, e^{k+1/2}) - \\ & \psi(e^{k+1/2}, e^{k+1/2}), \end{aligned}$$

然后利用引理 6, 可以得到

$$\begin{aligned} & -(\psi(U^{k+1/2}, U^{k+1/2}) - \psi(u^{k+1/2}, u^{k+1/2}), e^{k+1/2}) = \\ & -(\psi(e^{k+1/2}, U^{k+1/2}), e^{k+1/2}) = \\ & -h \sum_{i=1}^{n_1-1} [e_i^{k+1/2} \Delta_x U_i^{k+1/2} + \Delta_x (e_i^{k+1/2} U_i^{k+1/2})] \cdot \\ & e_i^{k+1/2} / 3 = -h \sum_{i=1}^{n_1-1} (e_i^{k+1/2})^2 \Delta_x U_i^{k+1/2} / 3 + \\ & h \sum_{i=1}^{n_1-1} e_i^{k+1/2} U_i^{k+1/2} \Delta_x e_i^{k+1/2} / 3 = \\ & -h \sum_{i=1}^{n_1-1} (e_i^{k+1/2})^2 \Delta_x U_i^{k+1/2} / 3 - \\ & h \sum_{i=0}^{n_1-1} (U_{i+1}^{k+1/2} - U_i^{k+1/2}) e_i^{k+1/2} e_{i+1}^{k+1/2} / 6 / h \leq \\ & c_1 \|e^{k+1/2}\|^2 / 2. \end{aligned} \quad (22)$$

将式 (22) 代入式 (21) 中, 并且利用 Cauchy-Schwarz 和引理 5 可得

$$\begin{aligned} & (\|e^{k+1}\|^2 - \|e^k\|^2) / 2\tau \leq c_1 \|e^{k+1/2}\|^2 / 2 + c_1 \|e^{k+1/2}\|^2 + \\ & ((R_1)^k, e^{k+1/2}) = 3c_1 \|e^{k+1/2}\|^2 / 2 + ((R_1)^k, e^{k+1/2}) \leq \\ & 3c_1 \|e^{k+1/2}\|^2 / 2 + \|(R_1)^k\| \cdot \|e^{k+1/2}\| \leq \\ & 3c_1 (\|e^k\| + \|e^{k+1}\| / 2) / 2 + \|(R_1)^k\| \cdot \|e^k\| + \|e^{k+1}\| / 2, \\ & 0 \leq k \leq n_2 - 1. \end{aligned}$$

以上式子两边同时约掉 $(\|e^k\| + \|e^{k+1}\|) / 2$, 得

$$\begin{aligned} & (\|e^{k+1}\| - \|e^k\|) / \tau \leq 3c_1 (\|e^{k+1}\| + \|e^k\|) / 2 + \|(R_1)^k\|, \\ & 0 \leq k \leq n_2 - 1, \end{aligned}$$

即

$$\begin{aligned} & (\|e^{k+1}\| - \|e^k\|) \leq 3c_1 \tau (\|e^{k+1}\| + \|e^k\|) / 2 + \tau \|(R_1)^k\|, \\ & 0 \leq k \leq n_2 - 1. \end{aligned}$$

也就是说,

$$\begin{aligned} & (1 - 3c_1 \tau / 2) \|e^{k+1}\| \leq (1 + 3c_1 \tau / 2) \|e^k\| + \tau \|(R_1)^k\|, \\ & 0 \leq k \leq n_2 - 1. \end{aligned}$$

当 $\tau < 2/3c_1$ 时, 由此得出,

$$\begin{aligned} & \|e^{k+1}\| \leq (1 + 6c_1 \tau) \|e^k\| + 2\tau \|(R_1)^k\| \leq (1 + 6c_1 \tau) \|e^k\| + \\ & 2c_2 \tau \sqrt{L_1} (\tau^2 + h^2), 0 \leq k \leq n_2 - 1. \end{aligned}$$

根据引理 2, 观察到 $\|e^0\| = 0$, 所以

$$\|e^k\| \leq e^{6c_1 \tau} \cdot \frac{c_2 \sqrt{L_1}}{3c_1} (\tau^2 + h^2) = c_3 (\tau^2 + h^2), 0 \leq k \leq n_2 - 1.$$

$$\text{其中 } c_3 = e^{6c_1 \tau} \cdot \frac{c_2 \sqrt{L_1}}{3c_1}.$$

5 数值算例

本节通过数值算例, 验证这篇文章建立的二层非线性差分格式的有效性。

算例 1

$$\begin{cases} u_t + uu_x = (x+t)u_{xx}, & 0 \leq x \leq 1, 0 < t \leq 1; \\ u(x, 0) = \sin(\pi x), & 0 < x < 1; \\ u(0, t) = 0, u(1, t) = 0, & 0 \leq t \leq 1. \end{cases}$$

从表 1 中可以看出, 随着空间剖分加密, 数值解的最大误差越来越小, 通过误差比可以看出, 该差分格式在空间上具有 2 阶精度。

从表 2 可以看出, 随着时间剖分加密, 数值解的最大误差越来越小, 通过误差比可以看出, 该差分格式在时间上具有 2 阶精度。

表 1 算例 1 取不同空间步长时数值解的最大误差和收敛阶
Table 1 Maximum error and convergence order of numerical solutions for different spatial step sizes in numerical examples 1

h	$E_\infty(h, \tau)$	$E_\infty(2h, \tau) / E_\infty(h, \tau)$
1/20	7.788 8e-04	*
1/40	1.971 4e-04	1.982 2
1/80	4.949 5e-05	1.993 8
1/160	1.239 1e-05	1.998 0
1/320	3.099 1e-06	1.999 3
1/640	7.748 7e-07	1.999 8

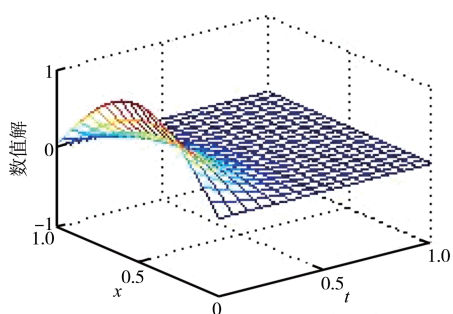
表2 算例1取不同时间步长时数值解的
最大误差和收敛阶

Table 2 Maximum error and convergence order of
numerical solutions for different time steps in the
calculation examples 1

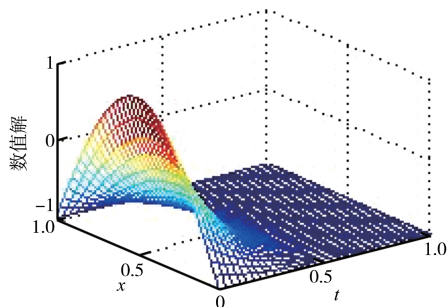
τ	$E_{\infty}(h, \tau)$	$E_{\infty}(h, 2\tau)/E_{\infty}(h, \tau)$
1/20	2.059 1e-03	*
1/40	5.126 6e-04	2.005 9
1/80	1.444 6e-04	1.827 3
1/160	4.235 2e-05	1.770 2
1/320	1.139 7e-05	1.893 8
1/640	2.900 8e-06	1.974 1

图1给出了不同时间步长和空间步长下的数值解。图2给出了 $n=64$ 时的空间收敛阶, 图3给出了 $m=64$ 时的时间收敛阶, 图4给出了 $t=1$ 时的数值解曲线。

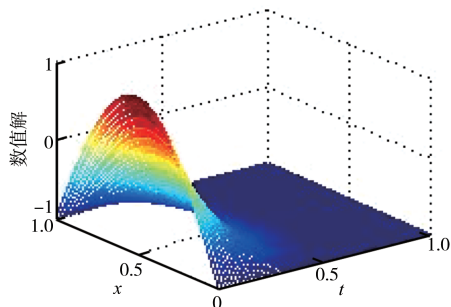
从图2和图3可以看出, 差分格式的空间、时间收敛阶是2。



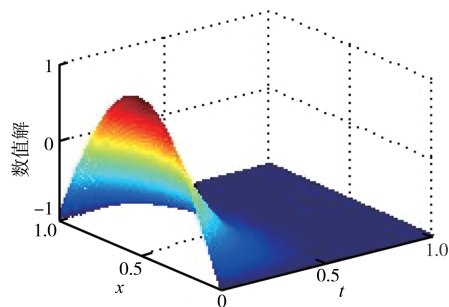
a) $h=1/20$ 和 $\tau=1/15$ 时的数值解



b) $h=1/40$ 和 $\tau=1/30$ 时的数值解



c) $h=1/80$ 和 $\tau=1/60$ 时的数值解



d) $h=1/160$ 和 $\tau=1/120$ 时的数值解

图1 算例取不同步长时精确解

Fig. 1 An illustration of asynchronous long-term exact solutions

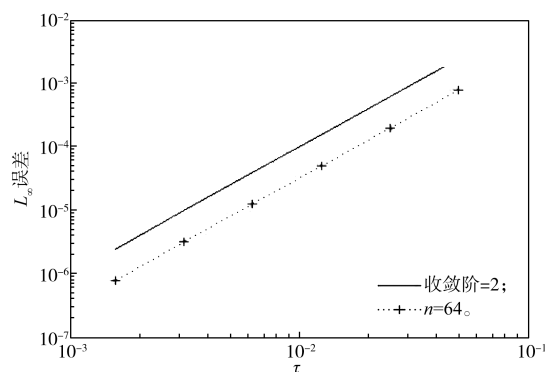


图2 $n=64$ 时空间收敛阶

Fig. 2 Space convergence order with $n=64$

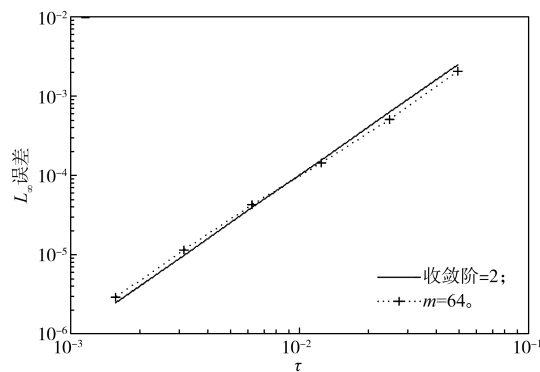


图3 $m=64$ 时时间收敛阶

Fig. 3 Time convergence order with $m=64$

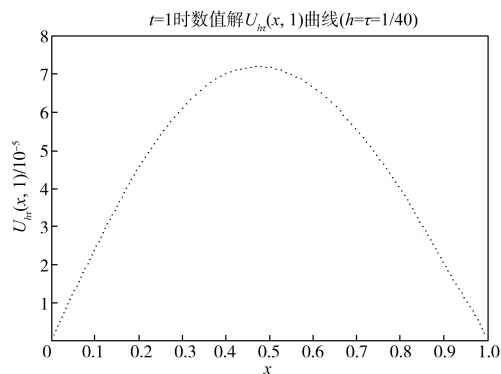


图4 $t=1$ 时数值解曲线

Fig. 4 Numerical solution curve with $t=1$

算例 2

$$\begin{cases} u_t + uu_x = (3x + 2t)u_{xx}, & 0 \leq x \leq 1, 0 < t \leq 1; \\ u(x, 0) = \sin(\pi x), & 0 < x < 1; \\ u(0, t) = 0, u(1, t) = 0, & 0 \leq t \leq 1. \end{cases}$$

其取不同空间步长和时间步长时的数值解最大误差和收敛分别见表 3 和表 4。

表 3 算例 2 取不同空间步长时的
数值解最大误差和收敛阶

Table 3 Maximum error and convergence order of
numerical solutions for different spatial step sizes in
numerical examples 2

h	$E_{\infty}(h, \tau)$	$E_{\infty}(2h, \tau) / E_{\infty}(h, \tau)$
1/20	6.435 6e-04	*
1/40	1.748 7e-04	1.879 7
1/80	4.526 6e-05	1.949 8
1/160	1.145 9e-05	1.982 0
1/320	2.876 5e-06	1.994 1
1/640	7.199 8e-07	1.998 3

从表 3 中可以看出, 随着空间剖分加密, 数值解的最大误差越来越小, 通过误差比可以看出, 该差分格式在空间上具有 2 阶精度。

表 4 算例 2 取不同时间步长时的数值解
最大误差和收敛阶

Table 4 Maximum error and convergence order of numerical
solutions for different time steps in the calculation examples 2

τ	$E_{\infty}(h, \tau)$	$E_{\infty}(h, 2\tau) / E_{\infty}(h, \tau)$
1/20	3.076 4e-02	*
1/40	8.169 9e-03	1.912 9
1/80	2.018 0e-03	2.017 4
1/160	4.903 3e-04	2.041 1
1/320	1.186 7e-04	2.046 8
1/640	2.891 5e-05	2.037 1

从表 4 中可以看出, 随着时间剖分加密, 数值解的最大误差越来越小, 通过误差比可以看出, 该差分格式在时间上具有 2 阶精度。

6 结语

本文研究了一维非线性变系数 Burgers 方程初边值问题的差分方法。首先, 建立了一个二层非线性差分格式, 然后证明了该差分格式具有稳定性、收敛性, 而且在空间和时间上都具有 2 阶精度, 最后用该差分格式计算了两个具体的算例, 结果表明该差分格式是有效的。

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