# 一类具有时滞和放养的扩散系统的周期解

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**摘 要**：研究一类具有 Beddington 功能反应和放养的 Lotka-Volterra 扩散系统。证明了系统正周期解的存在性，并通过构造适当的 Lyapunov 泛函，给出了正周期解全局稳定的充分条件。

**关键词**：时滞；扩散；放养；正周期解；全局稳定性

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## The Periodic Solutions of Diffusion System with Stock and Time Delay

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**Abstract**：Researches a Lotka-Volterra diffusion system with Beddington function response and stock. Proves the existence of positive periodic solution for the system, and by constructing appropriate Lyapunov function, provides sufficient conditions for the global stability of positive periodic solution.

**Keywords**：time delay；diffusion；stock；positive periodic solution；global stability

## 1 研究背景

国内外很多学者对具有功能反应的捕食周期系统进行了大量研究[1-8]。如：文献[6]研究了时标上具有阶段结构的三种群捕食系统。运用时标上连续拓扑度定理，得到了系统存在周期解的充分条件，其研究方法使系统的连续时间情形和离散时间情形的周期解问题得到了统一。文献[7]研究了一类具有保护区和避难所效应的捕食系统周期解存在性。利用重合度理论，得到系统至少存在 8 个正周期解的充分条件，并举例子说明了结果的有效性。由于环境的变化、人为的干预或外来物的影响，都不同程度地影响着生物的持续生存和灭绝，于是越来越多的现实因素被考虑到模型中来。不论是野生的还是人工饲养的种群，人们常采取扩散和放养（食饵补充）来加以保护或控制。特别是当捕食者种类多、密度大或捕食能力强时，常通过补充食饵使种群的数量趋于稳定，以达到生态平衡。

文献[8]研究了具有扩散和放养的两种群竞争系统，得到了系统周期解的存在性，唯一性和全局渐近稳定性的充分条件，但文献[8]研究的种群比较少，且没有给出捕食者种群的捕食率，而生态系统中多见的是多种群相继捕食的生物链。因此，本文在文献[8]的基础上考虑了捕食率为 Beddington 型功能反应的三种群相继捕食的扩散模型，即如下模型：

$$\begin{cases} x_1'(t) = x_1(t)\Big[r_1(t) - a_1(t)x_1(t) - \\ \qquad \dfrac{c_1(t)x_3(t)}{h(t) + x_1(t) + v(t)x_3(t)} - \\ \qquad \beta_1(t)\displaystyle\int_{-\tau}^{0} k_1(s)x_1(t+s)\mathrm{d}s\Big] + \\ \qquad D_1(t)[x_2(t) - x_1(t)] + S_1(t), \\ x_2'(t) = x_2(t)\Big[r_2(t) - a_2(t)x_2(t) - \\ \qquad \beta_2(t)\displaystyle\int_{-\tau}^{0} k_2(s)x_2(t+s)\mathrm{d}s\Big] + \\ \qquad D_2(t)[x_1(t) - x_2(t)] + S_2(t), \\ x_3'(t) = x_3(t)\Big[-r_3(t) - a_3(t)x_3(t) + \\ \qquad \dfrac{c_2(t)x_1(t-\tau)}{h(t) + x_1(t-\tau) + v(t)x_3(t-\tau)} - \\ \qquad \beta_3(t)\displaystyle\int_{-\tau}^{0} k_3(s)x_3(t+s)\mathrm{d}s - \\ \qquad \dfrac{c_3(t)x_4(t)}{h(t) + x_3(t) + v(t)x_4(t)}\Big], \\ x_4'(t) = x_4(t)\Big[-r_4(t) - a_4(t)x_4(t) - \\ \qquad \beta_4(t)\displaystyle\int_{-\tau}^{0} k_4(s)x_4(t+s)\mathrm{d}s + \\ \qquad \dfrac{c_4(t)x_3(t-\tau)}{h(t) + x_3(t-\tau) + v(t)x_4(t-\tau)}\Big]. \end{cases} \quad (1)$$

式中：$x_i(t)$（$i=1,2,3,4$）分别为被捕食种群和捕食者种群的密度，$x_1(t)$，$x_2(t)$ 分别为被捕食种群在斑块 I 和 II 中的密度，而捕食者种群被限制在斑块中不能扩散；

　　$x_3(t)$ 以 $x_1(t)$ 为食，$x_4(t)$ 以 $x_3(t)$ 为食，如此相继捕食，形成了一个生物食物链；

　　$D_i(t)$（$i=1,2$），$S_i(t)$（$i=1,2$）分别为被捕食种群在斑块 I 或 II 中的扩散率和投放率，

　　$r_i(t)$，$a_i(t)$，$c_i(t)$，$\beta_i(t)$（$i=1,2,3,4$）均为模型的系数；$k_i(s)$（$i=1,2,3,4$），$v(t)$，$h(t)$ 均表示函数。

## 2　基本概念和假设

对连续有界函数 $f(t)$，记 $\overline{f} = \dfrac{1}{\omega}\displaystyle\int_0^\omega f(t)\mathrm{d}t$，

$f^m = \sup\limits_{t\in[0,\omega]}\{f(t)\}, f^l = \inf\limits_{t\in[0,\omega]}\{f(t)\}$。

假设模型（1）的系数满足：

H1　$r_i(t)$，$a_i(t)$，$\beta_i(t)$，$c_i(t)$（$i=1,2,3,4$），$D_i(t)$，$S_i(t)$（$i=1,2$），$h(t)$，$v(t)$ 是连续的严格正周期函数，且满足

$$\min\{r_i^l, a_i^l, \beta_i^l, c_i^l, D_j^l, S_j^l, h^l,$$

$$v^l\ (i=1,2,3,4; j=1,2)\} > 0,$$

$$\max\{r_i^m, a_i^m, \beta^m, c_i^m, D_j^m, S_j^m, h^m,$$

$$v^m\ (i=1,2,3,4; j=1,2)\} < \infty。$$

H2　$k_i(s)$ 是分段连续函数，且满足 $\displaystyle\int_{-\tau}^{0} k_i(s)\mathrm{d}s = 1$，$k_i(s) \geqslant 0$，$s \in [-\tau, 0]$，$\tau \geqslant 0$。

令 $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \in \mathbf{R}^4$，

$$\mathbf{R}^4 = \{x \in \mathbf{R}^4 \mid x_i \geqslant 0, i = 1,2,3,4\},$$

$x \in \mathrm{int}\,\mathbf{R}_+^4$ 表示 $x>0$。

根据生物学的意义，仅在 $x \in \mathrm{int}\,\mathbf{R}_+^4$ 中讨论模型（1）。系统（1）的初始条件为

$$x_i(\theta) = \phi_i(\theta) > 0\ (i=1,2,3,4),\ \theta \in [-\tau, 0], \quad (2)$$

式中 $\phi_i(\theta)$ 表示非负的连续函数。

## 3　解的正性

**引理 1**　系统（1）满足初始条件（2）的解 $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ 必在区间 $[0, +\infty)$ 上存在，且对一切 $t \geqslant 0$，恒有 $x>0$。

**证**　对任意 $\varepsilon_i > 0$（$i=1,2,3,4$），构造如下方程组

$$\begin{cases} x_1'(t) = x_1(t)\Big[r_1(t) - a_1(t)x_1(t) - \\ \qquad \dfrac{c_1(t)x_3(t)}{h(t) + x_1(t) + v(t)x_3(t)} - \\ \qquad \beta_1(t)\displaystyle\int_{-\tau}^{0} k_1(s)x_1(t+s)\mathrm{d}s\Big] + \\ \qquad D_1(t)(x_2(t) - x_1(t)) + S_1(t) + \varepsilon_1, \\ x_2'(t) = x_2(t)\Big[r_2(t) - a_2(t)x_2(t) - \\ \qquad \beta_2(t)\displaystyle\int_{-\tau}^{0} k_2(s)x_2(t+s)\mathrm{d}s\Big] + \\ \qquad D_2(t)(x_1(t) - x_2(t)) + S_2(t) + \varepsilon_2, \\ x_3'(t) = x_3(t)\Big[-r_3(t) - a_3(t)x_3(t) + \\ \qquad \dfrac{c_2(t)x_1(t-\tau)}{h(t) + x_1(t-\tau) + v(t)x_3(t-\tau)} - \\ \qquad \beta_3(t)\displaystyle\int_{-\tau}^{0} k_3(s)x_3(t+s)\mathrm{d}s - \\ \qquad \dfrac{c_3(t)x_4(t)}{h(t) + x_3(t) + v(t)x_4(t)}\Big] + \varepsilon_3, \\ x_4'(t) = x_4(t)\Big[-r_4(t) - a_4(t)x_4(t) - \\ \qquad \beta_4(t)\displaystyle\int_{-\tau}^{0} k_4(s)x_4(t+s)\mathrm{d}s + \\ \qquad \dfrac{c_4(t)x_3(t-\tau)}{h(t) + x_3(t-\tau) + v(t)x_4(t-\tau)}\Big] + \varepsilon_4。 \end{cases} \quad (3)$$

设 $u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))$ 是方程组（3）满足初始条件（2）的解，则 $u(t)>0$。若不然，不仿设

存在$\rho>0$使$u_1(\rho)>0$，且对任意$t\in(0,\rho)$有$u_i(t)\geqslant 0(i=1,2,3,4)$，则

$$u_1'(\rho)=D_1(\rho)u_2(\rho)+\varepsilon_1\geqslant\varepsilon_1>0。$$

由函数的连续性可知，存在$\delta>0(\delta<\rho)$，使得对任意的$t\in\left[\rho-\delta,\rho+\delta\right]$有$u_1'(t)>0$，从而有$u_1\left(\rho-\dfrac{\delta}{2}\right)<u_1(\rho)=0$，矛盾，因此$u_1(t)>0$，$t\geqslant 0$。

同理$u_2(t)>0$，$u_3(t)>0$，$u_4(t)>0$，$t\geqslant 0$。

由于方程组（3）的解关于$\varepsilon_i(i=1,2,3,4)$是任意的，令$\varepsilon_i\rightarrow 0$（$i=1,2,3,4$），得系统（1）的解

$$x(t)=\left(x_1(t),x_2(t),x_3(t),x_4(t)\right)$$

满足$x_i'(t)\geqslant 0$（$i=1,2,3,4$），$t\geqslant 0$。于是由模型（1）及初始条件（2）可得引理1的结论。

## 4　周期解的持久性

**定义1**　若存在紧区域$D\subset\mathbf{R}_+^4$，对系统（1）满足初始值的任意解$x(t)=\left(x_1(t),x_2(t),x_3(t),x_4(t)\right)$，存在$T\geqslant 0$，使得当$t\geqslant T$时均有$x(t)\in D$，则称系统（1）是一致持久的。

**引理2**　设模型（1）满足条件H1和H2以及以下条件：

H3　$c_2^m>r_3^l$；

H4　$c_4^m>r_4^l$；

H5　$r_2^l>\beta_2^m M_1$；

H6　$\dfrac{c_2^{\,l}}{a_3^{\,m}\left(h^m+M_1+v^m M_2\right)}>$

$\left(\dfrac{r_3}{a_3}\right)^m+\left(\dfrac{\beta_3}{a_3}\right)^m M_2+\dfrac{1}{a_3^{\,m}}\left(\dfrac{c_3}{v}\right)^m$；

H7　$\dfrac{c_4^{\,l}}{a_4^{\,m}\left(h^m+M_2+v^m M_2\right)}>$

$\left(\dfrac{r_4}{a_4}\right)^m+\left(\dfrac{\beta_4}{a_4}\right)^m M_2$。

则存在$m_i>0, M_i>0(i=1,2)$，使得集合

$$D=\Big\{\left(x_1(t),x_2(t),x_3(t),x_4(t)\right)\in$$
$$\mathbf{R}_+^4\,\big|\,m_1\leqslant x_i(t)\leqslant M_1,m_2\leqslant x_j(t)\leqslant M_2;$$
$$i=1,2\,;\,j=3,4\Big\}$$

是模型（1）的最终有界集和正不变集。

**证**　对模型（1）作变换。

令$x_i(t)=\mathrm{e}^{u_i(t)}(i=1,2,3,4)$则模型（1）化为

$$\left\{\begin{aligned}
u_1'(t)&=r_1(t)-a_1(t)\mathrm{e}^{u_1(t)}-\\
&\quad\frac{c_1(t)\mathrm{e}^{u_3(t)}}{h(t)+\mathrm{e}^{u_1(t)}+v(t)\mathrm{e}^{u_3(t)}}-\\
&\quad\beta_1(t)\int_{-\tau}^0 k_1(s)\mathrm{e}^{u_1(t+s)}\mathrm{d}s+\\
&\quad D_1(t)\mathrm{e}^{u_2(t)-u_1(t)}-D_1(t)+S_1(t)\mathrm{e}^{-u_1(t)},\\
u_2'(t)&=r_2(t)-a_2(t)\mathrm{e}^{u_2(t)}-\\
&\quad\beta_2(t)\int_{-\tau}^0 k_2(s)\mathrm{e}^{u_2(t+s)}\mathrm{d}s+\\
&\quad D_2(t)\mathrm{e}^{u_1(t)-u_2(t)}-D_2(t)+S_2(t)\mathrm{e}^{-u_2(t)},\\
u_3'(t)&=-r_3(t)-a_3(t)\mathrm{e}^{u_3(t)}+\\
&\quad\frac{c_2(t)\mathrm{e}^{u_1(t-\tau)}}{h(t)+\mathrm{e}^{u_1(t-\tau)}+v(t)\mathrm{e}^{u_3(t-\tau)}}-\\
&\quad\beta_3(t)\int_{-\tau}^0 k_3(s)\mathrm{e}^{u_3(t-\tau)}\mathrm{d}s-\\
&\quad\frac{c_3(t)\mathrm{e}^{u_4(t)}}{h(t)+\mathrm{e}^{u_3(t)}+v(t)\mathrm{e}^{u_4(t)}},\\
u_4'(t)&=-r_4(t)-a_4(t)\mathrm{e}^{u_4(t)}-\\
&\quad\beta_4(t)\int_{-\tau}^0 k_4(s)\mathrm{e}^{u_4(t+s)}\mathrm{d}s+\\
&\quad\frac{c_4(t)\mathrm{e}^{u_3(t-\tau)}}{h(t)+\mathrm{e}^{u_3(t-\tau)}+v(t)\mathrm{e}^{u_4(t-\tau)}}。
\end{aligned}\right.\tag{4}$$

选取$t_i,\tau_i\in[0,\omega](i=1,2,3,4)$使得

$$\ln x_i(t_i)=u_i(t_i)=\max_{t\in[0,\omega]}u_i(t)\quad(i=1,2,3,4),$$
$$\ln x_i(\tau_i)=u_i(\tau_i)=\min_{t\in[0,\omega]}u_i(t)\quad(i=1,2,3,4)。$$

于是有：

$$r_1(t_1)-a_1(t_1)\mathrm{e}^{u_1(t_1)}-\frac{c_1(t_1)\mathrm{e}^{u_3(t_1)}}{h(t_1)+\mathrm{e}^{u_1(t_1)}+v(t_1)\mathrm{e}^{u_3(t_1)}}-$$
$$\beta_1(t_1)\int_{-\tau}^0 k_1(s)\mathrm{e}^{u_1(t_1+s)}\mathrm{d}s-D_1(t_1)+$$
$$D_1(t_1)\mathrm{e}^{u_2(t_1)-u_1(t_1)}+S_1(t_1)\mathrm{e}^{-u_1(t_1)}=0,\tag{5}$$

$$r_2(t_2)-a_2(t_2)\mathrm{e}^{u_2(t_2)}-\beta_2(t_2)\int_{-\tau}^0 k_2(s)\mathrm{e}^{u_2(t_2+s)}\mathrm{d}s+$$
$$D_2(t_2)\mathrm{e}^{u_1(t_2)-u_2(t_2)}-D_2(t_2)+S_2(t_2)\mathrm{e}^{-u_2(t_2)}=0,\tag{6}$$

$$-r_3(t_3)-a_3(t_3)\mathrm{e}^{u_3(t_3)}+\frac{c_2(t_3)\mathrm{e}^{u_1(t_3-\tau)}}{h(t_3)+\mathrm{e}^{u_1(t_3-\tau)}+v(t_3)\mathrm{e}^{u_3(t_3-\tau)}}-$$
$$\beta_3(t_3)\int_{-\tau}^0 k_3(s)\mathrm{e}^{u_3(t_3-\tau)}\mathrm{d}s-$$
$$\frac{c_3(t_3)\mathrm{e}^{u_4(t_3)}}{h(t_3)+\mathrm{e}^{u_3(t_3)}+v(t_3)\mathrm{e}^{u_4(t_3)}}=0,\tag{7}$$

$$-r_4(t_4)-a_4(t_4)\mathrm{e}^{u_4(t_4)}-\beta_4(t_4)\int_{-\tau}^0 k_4(s)\mathrm{e}^{u_4(t_4+s)}\mathrm{d}s+$$
$$\frac{c_4(t_4)\mathrm{e}^{u_3(t_4-\tau)}}{h(t_4)+\mathrm{e}^{u_3(t_4-\tau)}+v(t_4)\mathrm{e}^{u_4(t_4-\tau)}}=0,\tag{8}$$

$$r_1(\tau_1) - a_1(\tau_1)e^{u_1(\tau_1)} - \frac{c_1(\tau_1)e^{u_3(\tau_1)}}{h(\tau_1) + e^{u_1(\tau_1)} + v(\tau_1)e^{u_3(\tau_1)}} -$$

$$\beta_1(\tau_1)\int_{-\tau}^{0}k_1(s)e^{u_1(\tau_1+s)}ds - D_1(\tau_1) +$$

$$D_1(\tau_1)e^{u_2(\tau_1)-u_1(\tau_1)} + S_1(\tau_1)e^{-u_1(\tau_1)} = 0, \qquad (9)$$

$$r_2(\tau_2) - a_2(\tau_2)e^{u_2(\tau_2)} - \beta_2(\tau_2)\int_{-\tau}^{0}k_2(s)e^{u_2(\tau_2+s)}ds +$$

$$D_2(\tau_2)e^{u_1(\tau_2)-u_2(\tau_2)} - D_2(\tau_2) + S_2(\tau_2)e^{-u_2(\tau_2)} = 0, \qquad (10)$$

$$-r_3(\tau_3) - a_3(\tau_3)e^{u_3(\tau_3)} + \frac{c_2(\tau_3)e^{u_1(\tau_3-\tau)}}{h(\tau_3) + e^{u_1(\tau_3-\tau)} + v(\tau_3)e^{u_3(\tau_3-\tau)}} -$$

$$\beta_3(\tau_3)\int_{-\tau}^{0}k_3(s)e^{u_3(\tau_3-\tau)}ds -$$

$$\frac{c_3(\tau_3)e^{u_4(\tau_3)}}{h(\tau_3) + e^{u_3(\tau_3)} + v(\tau_3)e^{u_4(\tau_3)}} = 0, \qquad (11)$$

$$-r_4(\tau_4) - a_4(\tau_4)e^{u_4(\tau_4)} - \beta_4(\tau_4)\int_{-\tau}^{0}k_4(s)e^{u_4(\tau_4+s)}ds +$$

$$\frac{c_4(\tau_4)e^{u_3(\tau_4-\tau)}}{h(\tau_4) + e^{u_3(\tau_4-\tau)} + v(\tau_4)e^{u_4(\tau_4-\tau)}} = 0. \qquad (12)$$

下面分情况讨论模型解的上界。

1）若 $u_1(t_1) \geqslant u_2(t_2)$，则显然有 $u_1(t_1) \geqslant u_2(t_1)$，因此由式（5）得

$$a_1(t_1)e^{u_1(t_1)} \leqslant r_1(t_1) + S_1(t_1)e^{-u_1(t_1)}.$$

而 $e^{u_1(t_1)} \leqslant \dfrac{r_1^m + \sqrt{(r_1^m)^2 + 4a_1^l S_1^m}}{2a_1^l}$，从而得

$$x_2(t_2) \leqslant x_1(t_1) \leqslant \frac{r_1^m + \sqrt{(r_1^m)^2 + 4a_1^l S_1^m}}{2a_1^l}.$$

2）若 $u_1(t_1) < u_2(t_2)$，则显然有 $u_1(t_2) < u_2(t_2)$，再由式（6）得

$$a_2(t_2)e^{u_2(t_2)} \leqslant r_2(t_2) + S_2(t_2)e^{-u_2(t_2)}.$$

同理得

$$x_1(t_1) \leqslant x_2(t_2) \leqslant \frac{r_2^m + \sqrt{(r_2^m)^2 + 4a_2^l S_2^m}}{2a_2^l}.$$

令 $M_j^* = \dfrac{r_j^m + \sqrt{(r_j^m)^2 + 4a_j^l S_j^m}}{2a_j^l}$ $(j=1,2)$，综上可得

$$x_i(t_i) = e^{u_i(t_i)} \leqslant \max\{M_1^*, M_2^*\} = M_1, \quad i = 1,2.$$

由式（7）有

$$a_3(t_3)e^{u_3(t_3)} < -r_3(t_3) + c_2(t_3),$$

再由 H3 有

$$x_3(t_3) < \frac{c_2^m - r_3^l}{a_3^l}.$$

同理由 H4 可得

$$x_4(t_4) < \frac{c_4^m - r_4^l}{a_4^l}.$$

令 $M_3^* = \dfrac{c_2^m - r_3^l}{a_3^l}$，$M_4^* = \dfrac{c_4^m - r_4^l}{a_4^l}$。

综上可得

$$\begin{cases} x_3(t_3) = e^{u_3(t_3)} < M_2, \\ x_4(t_4) = e^{u_4(t_4)} < M_2, \\ M_2 = \max\{M_3^*, M_4^*\}. \end{cases}$$

下面再分情况讨论模型解的下界。

1）若 $u_1(\tau_1) \leqslant u_2(\tau_2)$，则显然有 $u_1(\tau_1) \leqslant u_2(\tau_1)$，因此由式（6）得

$$r_1(\tau_1) - a_1(\tau_1)e^{u_1(\tau_1)} - \frac{c_1(\tau_1)e^{u_3(\tau_1)}}{h(\tau_1) + e^{u_1(\tau_1)} + v(\tau_1)e^{u_3(\tau_1)}} -$$

$$\beta_1(\tau_1)M_1 - D_1(\tau_1) + S_1(\tau_1)e^{-u_1(\tau_1)} +$$

$$D_1(\tau_1)e^{u_2(\tau_1)-u_1(\tau_1)} < 0,$$

所以有

$$a_1^m e^{u_1(\tau_1)} \geqslant r_1^l - \left(\frac{c_1}{v}\right)^m - \beta_1^m M_1 + S_1^l e^{-u_1(\tau_1)}.$$

将上式两端乘以 $e^{u_1(\tau_1)}$ 得

$$a_1^m e^{2u_1(\tau_1)} - \left(r_1^l - \left(\frac{c_1}{v}\right)^m - \beta_1^m M_1\right)e^{u_1(\tau_1)} - S_1^l \geqslant 0.$$

所以有

$$e^{u_1(\tau_1)} \geqslant$$

$$\frac{r_1^l - \left(\dfrac{c_1}{v}\right)^m - \beta_1^m M_1 + \sqrt{\left(r_1^l - \left(\dfrac{c_1}{v}\right)^m - \beta_1^m M_1\right)^2 + 4a_1^m S_1^l}}{2a_1^m},$$

即有

$$x_2(\tau_2) \geqslant x_1(\tau_1) \geqslant$$

$$\frac{r_1^l - \left(\dfrac{c_1}{v}\right)^m - \beta_1^m M_1 + \sqrt{\left(r_1^l - \left(\dfrac{c_1}{v}\right)^m - \beta_1^m M_1\right)^2 + 4a_1^m S_1^l}}{2a_1^m}.$$

2）若 $u_1(\tau_1) > u_2(\tau_2)$，则显然有 $u_1(\tau_2) > u_2(\tau_2)$，再由式（10）和 H5 得

$$a_2(\tau_2)e^{u_2(\tau_2)} \geqslant r_2(\tau_2) - \beta_2(\tau_2)M_1 + S_2(\tau_2)e^{-u_2(\tau_2)} \geqslant$$

$$r_2^l - \beta_2^m M_1.$$

因此有 $e^{u_2(\tau_2)} \geqslant \dfrac{r_2^l - \beta_2^m M_1}{a_2^m}$，即有

$$x_1(\tau_1) > x_2(\tau_2) \geqslant \frac{r_2^l - \beta_2^m M_1}{a_2^m}.$$

令

$m_1^* =$

$$\frac{r_1^l - \left(\frac{c_1}{v}\right)^m - \beta_1^m M_1 + \sqrt{\left(r_1^l - \left(\frac{c_1}{v}\right)^m - \beta_1^m M_1\right)^2 + 4a_1^m S_1^l}}{2a_1^m},$$

$$m_2^* = \frac{r_2^l - \beta_2^m M_1}{a_2^m}。$$

综上可得

$$x_i(\tau_i) = e^{u_i(\tau_i)} \geqslant \min\{m_1^*, m_2^*\} = m_1 \ (i=1,2)。$$

再由式（11）和H6得

$$a_3(\tau_3)e^{u_3(\tau_3)} > -r_3(\tau_3) + \frac{c_2(\tau_3)e^{u_1(\tau_3-\tau)}}{h(\tau_3) + e^{u_1(\tau_3-\tau)} + v(\tau_3)e^{u_3(\tau_3-\tau)}} -$$

$$\beta_3(\tau_3)M_2 - \frac{c_3(\tau_3)e^{u_4(\tau_3)}}{h(\tau_3) + e^{u_3(\tau_3)} + v(\tau_3)e^{u_4(\tau_3)}},$$

$$a_3(\tau_3)e^{u_3(\tau_3)} > -r_3^m + \frac{c_2^l}{h^m + M_1 + v^m M_2} - \beta_3^m M_2 - \left(\frac{c_3}{v}\right)^m。$$

所以有

$$e^{u_3(\tau_3)} > -\left(\frac{r_3}{a_3}\right)^m + \frac{c_2^l}{a_3^m(h^m + M_1 + v^m M_2)} -$$

$$\left(\frac{\beta_3}{a_3}\right)^m M_2 - \frac{1}{a_3^m}\left(\frac{c_3}{v}\right)^m \equiv m_3^*。$$

再由式（11）和H7可得

$$e^{u_3(\tau_3)} > -\left(\frac{r_3}{a_3}\right)^m + \frac{c_2^l}{a_3^m(h^m + M_1 + v^m M_2)} -$$

$$\left(\frac{\beta_3}{a_3}\right)^m M_2 - \frac{1}{a_3^m}\left(\frac{c_3}{v}\right)^m \equiv m_3^*。$$

令

$$m_3^* = \frac{c_2^l}{a_3^m(h^m + M_1 + v^m M_2)} - \left(\frac{r_3}{a_3}\right)^m -$$

$$\frac{\beta_3^m}{a_3^m}M_2 - \frac{1}{a_3^m}\left(\frac{c_3}{v}\right)^m,$$

$$m_4^* = \frac{c_4^l}{a_4^m(h^m + M_2 + v^m M_2)} - \left(\frac{r_4}{a_4}\right)^m - \frac{\beta_4^m}{a_4^m}M_2。$$

综上可得

$$\begin{cases} x_3(\tau_3) = e^{u_3(\tau_3)} \geqslant m_3^*, \\ x_4(\tau_4) = e^{u_4(\tau_4)} \geqslant m_4^*, \\ m_2 = \min\{m_3^*, m_4^*\}。 \end{cases}$$

由定义1和引理2可得如下定理：

**定理1** 如果模型（1）满足条件H1~H7，则模型（1）是一致持久的。

## 5 周期解的唯一性和全局稳定性

**定理2** 假设在H1~H7下，还有：

H8　　$a_4^l > \beta_4^m + \dfrac{c_3^m}{h^l + m_2 + v^l m_2}$；

H9　　$a_2^l > \beta_2^m + \dfrac{D_1^m}{m_1}$；

H10　　$\dfrac{c_2^m}{h^l + m_1 + v^l m_2} < a_1^l - \beta_1^m - \dfrac{D_2^M}{m_1}$；

H11　　$c_4^m < \left(a_3^l - \beta_3^m - \dfrac{c_1^m}{h^l + m_1 + v^l m_2}\right) \cdot$

$$\left(h^l + m_2 + v^l m_2\right)。$$

则模型（1）的正周期解是全局渐近稳定的，且是唯一的。

**证** 设 $(x_1(t), x_2(t), x_3(t), x_4(t))^T$ 是系统的任一正解，由于 $x_i(t) = e^{u_i(t)}\,(i=1,2,3,4)$，在这里假设 $(u_1(t), u_2(t), u_3(t), u_4(t))^T$ 是模型（1）的正周期解。

选取 Lyapunov 泛函如下

$$V(t) = \sum_{i=1}^4 \ln|x_i(t) - u_i(t)| +$$

$$\sum_{i=1}^4 \beta_i^M \int_{-\tau}^0 k_i(s) \int_{t+s}^t |x_i(\theta) - u_i(\theta)| \mathrm{d}\theta \mathrm{d}s +$$

$$\sigma_1 \int_{t-\tau}^t \left| \frac{x_1(s)}{h(t)+x_1(s)+v(t)x_1(s)} - \frac{u_1(s)}{h(t)+u_1(s)+v(t)u_1(s)} \right| \mathrm{d}s +$$

$$\sigma_2 \int_{t-\tau}^t \left| \frac{x_3(s)}{h(t)+x_3(s)+v(t)x_4(s)} - \frac{u_3(s)}{h(t)+u_3(s)+v(t)u_4(s)} \right| \mathrm{d}s$$

式中 $\sigma_1$，$\sigma_2$ 为常数，且满足

$$\begin{cases} \dfrac{c_2^m}{h^l + m_1 + v^l m_2} < \sigma_1 < a_1^l - \beta_1^m - \dfrac{D_2^M}{m_1}, \\ c_4^m < \sigma_2 < \left(a_3^l - \beta_3^m - \dfrac{c_1^m}{h^l + m_1 + v^l m_2}\right)\left(h^l + m_2 + v^l m_2\right)。 \end{cases}$$

（13）

按模型（1）计算 $V(t)$ 的右上导数得

$$V'(t) \leqslant -a_1^l|x_1(t) - u_1(t)| +$$

$$\frac{c_1^M}{h^L + e^{m_1} + v^L e^{m_2}}|x_3(t) - u_3(t)| +$$

$$\beta_1^M \int_{-\tau}^0 k_1(s)|x_1(t+s) - u_1(t+s)| \mathrm{d}s +$$

$$\tilde{D}_1 + \tilde{S}_1 - a_2^l|x_2(t) - u_2(t)| +$$

$$\beta_2^M \int_{-\tau}^0 k_2(s)|x_2(t+s) - u_2(t-\tau)| \mathrm{d}s + \tilde{D}_2 + \tilde{S}_2 -$$

$$a_3^L|x_3(t) - u_3(t)| + \frac{c_2^M}{h^L + e^{m_1} + v^L e^{m_2}}|x_1(t-\tau) -$$

$$u_2(t+s)| + \beta_3^M \int_{-\tau}^0 k_3(s)|x_3(t+s) - u_3(t+s)| \mathrm{d}s +$$

$$\frac{c_3^M}{h^L + e^{m_2} + v^L e^{m_2}}|x_4(t) - u_4(t)| - a_4^L|x_4(t) - u_4(t)| +$$

$$\beta_4^M \int_{-\tau}^0 k_4(s)|x_4(t+s) - u_4(t+s)| \mathrm{d}s +$$

$$\frac{c_4^M}{h^L+\mathrm{e}^{m_2}+v^L\mathrm{e}^{m_2}}\left|x_3\left(t-\tau\right)-u_3\left(t-\tau\right)\right|+$$

$$\beta_1^M\left|x_1\left(t\right)-u_1\left(t\right)\right|+\beta_2^M\left|x_2\left(t\right)-u_2\left(t\right)\right|+$$

$$\beta_3^M\left|x_3\left(t\right)-u_3\left(t\right)\right|+\beta_4^M\left|x_4\left(t\right)-u_4\left(t\right)\right|-$$

$$\beta_1^M\int_{-\tau}^0 k_1\left(s\right)\left|x_1\left(t+s\right)-u_1\left(t+s\right)\right|\mathrm{d}s-$$

$$\beta_2^M\int_{-\tau}^0 k_2\left(s\right)\left|x_2\left(t+s\right)-u_2\left(t+s\right)\right|\mathrm{d}s-$$

$$\beta_3^M\int_{-\tau}^0 k_3\left(s\right)\left|x_3\left(t+s\right)-u_3\left(t+s\right)\right|\mathrm{d}s-$$

$$\beta_4^M\int_{-\tau}^0 k_4\left(s\right)\left|x_4\left(t+s\right)-u_4\left(t+s\right)\right|\mathrm{d}s+$$

$$\frac{\sigma_1}{h^L+\mathrm{e}^{m_1}+v^L\mathrm{e}^{m_2}}\left|x_1\left(t\right)-u_1\left(t\right)\right|-$$

$$\frac{\sigma_1}{h^L+\mathrm{e}^{m_1}+v^L\mathrm{e}^{m_2}}\left|x_1\left(t-\tau\right)-u_1\left(t-\tau\right)\right|+$$

$$\frac{\sigma_2}{h^L+\mathrm{e}^{m_2}+v^L\mathrm{e}^{m_2}}\left|x_3\left(t\right)-u_3\left(t\right)\right|-$$

$$\frac{\sigma_2}{h^L+\mathrm{e}^{m_2}+v^L\mathrm{e}^{m_2}}\left|x_3\left(t-\tau\right)-u_3\left(t-\tau\right)\right|=$$

$$-\left(a_1^L-\beta_1^M-\frac{\sigma_1}{h^L+\mathrm{e}^{m_1}+v^L\mathrm{e}^{m_2}}\right)\left|x_1\left(t\right)-u_1\left(t\right)\right|-$$

$$\left(a_2^L-\beta_2^M\right)\left|x_2\left(t\right)-u_2\left(t\right)\right|-$$

$$\left(a_3^L-\beta_3^M-\frac{\sigma_2}{h^L+\mathrm{e}^{m_2}+v^L\mathrm{e}^{m_2}}-\frac{c_1^M}{h^L+\mathrm{e}^{m_1}+v^L\mathrm{e}^{m_2}}\right)\cdot$$

$$\left|x_3\left(t\right)-u_3\left(t\right)\right|-\left(a_4^L-\frac{c_3^M}{h^L+\mathrm{e}^{m_2}+v^L\mathrm{e}^{m_2}}-\beta_4^M\right)\cdot$$

$$\left|x_4\left(t\right)-u_4\left(t\right)\right|-\left(\frac{\sigma_1}{h^L+\mathrm{e}^{m_1}+v^L\mathrm{e}^{m_2}}-\frac{c_2^M}{h^L+\mathrm{e}^{m_1}+v^L\mathrm{e}^{m_2}}\right)\cdot$$

$$\left|x_1\left(t-\tau\right)-u_1\left(t-\tau\right)\right|-\left(\frac{\sigma_2}{h^L+\mathrm{e}^{m_2}+v^L\mathrm{e}^{m_2}}-\right.$$

$$\left.\frac{c_4^M}{h^L+\mathrm{e}^{m_2}+v^L\mathrm{e}^{m_2}}\right)\left|x_3\left(t-\tau\right)-u_3\left(t-\tau\right)\right|+$$

$$\tilde{D}_1+\tilde{S}_1+\tilde{D}_2+\tilde{S}_2,$$

$$（14）$$

式中：$\tilde{D}_1\left(t\right)=D_1\left(t\right)\left(\dfrac{x_2\left(t\right)}{x_1\left(t\right)}-\dfrac{u_2\left(t\right)}{u_1\left(t\right)}\right)\mathrm{sign}\left(x_1\left(t\right)-u_1\left(t\right)\right);$

$$\tilde{D}_2\left(t\right)=D_2\left(t\right)\left(\dfrac{x_1\left(t\right)}{x_2\left(t\right)}-\dfrac{u_1\left(t\right)}{u_2\left(t\right)}\right)\mathrm{sign}\left(x_2\left(t\right)-u_2\left(t\right)\right);$$

$$\tilde{S}_1\left(t\right)=\left(\dfrac{S_1\left(t\right)}{x_1\left(t\right)}-\dfrac{S_1\left(t\right)}{u_1\left(t\right)}\right)\mathrm{sign}\left(x_1\left(t\right)-u_1\left(t\right)\right);$$

$$\bar{S}_2\left(t\right)=D_2\left(t\right)\left(\dfrac{S_2\left(t\right)}{x_2\left(t\right)}-\dfrac{S_2\left(t\right)}{u_2\left(t\right)}\right)\mathrm{sig}\left(x_2\left(t\right)-u_2\left(t\right)\right)。$$

下面分情况讨论 $\tilde{D}_1\left(t\right)$ 和 $\tilde{S}_1\left(t\right)$。

1）当 $x_1(t)>u_1(t)$ 时

$$\tilde{D}_1\left(t\right)=D_1\left(t\right)\left(\dfrac{x_2\left(t\right)}{x_1\left(t\right)}-\dfrac{u_2\left(t\right)}{u_1\left(t\right)}\right)\mathrm{sign}\left(x_1\left(t\right)-u_1\left(t\right)\right)=$$

$$D_1\left(t\right)\left|\dfrac{x_2\left(t\right)}{x_1\left(t\right)}-\dfrac{u_2\left(t\right)}{u_1\left(t\right)}\right|\leqslant\dfrac{D_1\left(t\right)}{u_1\left(t\right)}\left|x_2\left(t\right)-u_2\left(t\right)\right|\leqslant$$

$$\dfrac{D_1^m}{m_1}\left|x_2\left(t\right)-u_2\left(t\right)\right|,$$

$$\tilde{S}_1\left(t\right)=\left(\dfrac{S_1\left(t\right)}{x_1\left(t\right)}-\dfrac{S_1\left(t\right)}{u_1\left(t\right)}\right)\mathrm{sign}\left(x_1\left(t\right)-u_1\left(t\right)\right)=$$

$$\dfrac{S_1\left(t\right)}{x_1\left(t\right)}-\dfrac{S_1\left(t\right)}{u_1\left(t\right)}<\dfrac{S_1\left(t\right)}{u_1\left(t\right)}-\dfrac{S_1\left(t\right)}{u_1\left(t\right)}=0。$$

2）当 $x_1(t)\leqslant u_1(t)$ 时

$$\tilde{D}_1\left(t\right)=D_1\left(t\right)\left|\dfrac{x_2\left(t\right)}{x_1\left(t\right)}-\dfrac{u_2\left(t\right)}{u_1\left(t\right)}\right|\leqslant\dfrac{D_1\left(t\right)}{x_1\left(t\right)}\left|x_2\left(t\right)-u_2\left(t\right)\right|\leqslant$$

$$\dfrac{D_1^m}{m_1}\left|x_2\left(t\right)-u_2\left(t\right)\right|,$$

$$\tilde{S}_1\left(t\right)=\left(\dfrac{S_1\left(t\right)}{x_1\left(t\right)}-\dfrac{S_1\left(t\right)}{u_1\left(t\right)}\right)\mathrm{sign}\left(x_1\left(t\right)-u_1\left(t\right)\right)=$$

$$\dfrac{S_1\left(t\right)}{u_1\left(t\right)}-\dfrac{S_1\left(t\right)}{x_1\left(t\right)}<\dfrac{S_1\left(t\right)}{x_1\left(t\right)}-\dfrac{S_1\left(t\right)}{u_1\left(t\right)}=0。$$

综上可得

$$\tilde{D}_1\left(t\right)\leqslant\dfrac{D_1^m}{m_1}\left|x_2\left(t\right)-u_2\left(t\right)\right|,\ \tilde{S}_1\left(t\right)=0。\qquad（15）$$

同理可得

$$\tilde{D}_2\left(t\right)\leqslant\dfrac{D_2^m}{m_1}\left|x_1\left(t\right)-u_1\left(t\right)\right|,\ \tilde{S}_2\left(t\right)\leqslant0。\qquad（16）$$

把式（15）和式（16）代入式（14）可得

$$V'\left(t\right)\leqslant-\left(a_1^l-\beta_1^m-\dfrac{D_2^m}{m_1}-\dfrac{\sigma_1}{h^l+m_1+v^lm_1}\right)\left|x_1\left(t\right)-u_1\left(t\right)\right|-$$

$$\left(a_2^l-\beta_2^m-\dfrac{D_1^m}{m_1}\right)\left|x_2\left(t\right)-u_2\left(t\right)\right|-$$

$$\left(a_3^l-\beta_3^m-\dfrac{\sigma_2}{h^l+m_2+v^lm_2}-\dfrac{c_1^m}{h^l+m_1+v^lm_2}\right)\cdot$$

$$\left|x_3\left(t\right)-u_3\left(t\right)\right|-\left(a_4^l-\dfrac{c_3^m}{h^l+m_2+v^lm_2}-\beta_4^m\right)\cdot$$

$$\left|x_4\left(t\right)-u_4\left(t\right)\right|-\left(\dfrac{\sigma_1}{h^l+m_1+v^lm_1}-\dfrac{c_2^m}{h^l+m_1+v^lm_2}\right)\cdot$$

$$\left|x_1\left(t-\tau\right)-u_1\left(t-\tau\right)\right|-$$

$$\left(\dfrac{\sigma_2}{h^l+m_2+v^lm_2}-\dfrac{c_4^m}{h^l+m_2+v^lm_2}\right)\cdot$$

$$\left|y_1\left(t-\tau\right)-v_1\left(t-\tau\right)\right|。$$

$$（17）$$

将式（13）代入式（17）可得

$$V'\left(t\right)\leqslant-\left(a_1^l-\beta_1^m-\dfrac{D_2^m}{m_1}-\dfrac{\sigma_1}{h^l+m_1+v^lm_1}\right).$$

$$\left|x_1(t)-u_1(t)\right|-\left(a_2^l-\beta_2^m-\frac{D_1^m}{m_1}\right)\left|x_2(t)-u_2(t)\right|-$$

$$\left(a_3^l-\beta_3^m-\frac{\sigma_2}{h^l+m_2+v^lm_2}-\frac{c_1^m}{h^l+m_1+v^lm_2}\right)\cdot$$

$$\left|x_3(t)-u_3(t)\right|-\left(a_4^l-\frac{c_3^m}{h^l+m_2+v^lm_2}-\beta_4^m\right)\cdot$$

$$\left|x_4(t)-u_4(t)\right|\circ$$

$$(18)$$

根据条件 H8~ H11，并令

$$\gamma=\min\left\{a_1^l-\beta_1^m-\frac{D_2^m}{m_1}-\frac{\sigma_1}{h^l+m_1+v^lm_1},\right.$$

$$a_2^l-\beta_2^m-\frac{D_1^m}{m_1},a_3^l-\beta_3^m-\frac{\sigma_2}{h^l+m_2+v^lm_2}-$$

$$\left.\frac{c_1^m}{h^l+m_1+v^lm_2},a_4^l-\frac{c_3^m}{h^l+m_2+v^lm_2}-\beta_4^m\right\}\circ\quad(19)$$

将式（19）代入式（18）得

$$V'(t)\leqslant-\gamma\sum_{i=1}^{4}\left|x_i(t)-u_i(t)\right|\circ$$

从 0 到 $t$ 对上式积分得

$$V(t)+\gamma\int_0^t\sum_{i=1}^{4}\left|x_i(s)-u_i(s)\right|\mathrm{d}s\leqslant V(0)<+\infty,$$

从而

$$\sum_{i=1}^{4}\left|x_i(t)-u_i(t)\right|\in L^1(0,+\infty)\circ$$

由 Barbalart 引理[9]得

$$\lim_{t\to+\infty}\sum_{i=1}^{4}\left|x_i(t)-u_i(t)\right|=0,$$

从而

$$\lim_{t\to\infty}\left(\left|x_i(t)-u_i(t)\right|\right)=0\ (i=1,2,3,4)\circ$$

因此，模型（1）的周期解是唯一的且是全局稳定的。

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