关于曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$的曲率和挠率的计算

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摘　要：在基本向量为$\vec{\alpha}$、$\vec{\beta}$、$\vec{\gamma}$且曲率和挠率分别为κ、τ的已知曲线$\Gamma:\vec{r}=\vec{r}(s)$的基础上，对曲率$\kappa$、挠率$\tau$分别取不同值的4种情形时，研究由$\vec{\alpha}$和$\vec{\beta}$所作出的曲线$\bar{\Gamma}:\bar{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$的曲率$\bar{\kappa}$和挠率$\bar{\tau}$的计算问题。

关键词：曲线；曲率；挠率；基本向量

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On the Calculation of the Curvature and Torsion of Curves $\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$

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Abstract：Based on the curve $\Gamma:\vec{r}=\vec{r}(s)$, whose basic vectors are $\vec{\alpha}$、$\vec{\beta}$、$\vec{\gamma}$ and curvature and torsion are κ and τ respectively and under four different conditions which κ and τ have different values, studies the calculation of curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ of curves $\bar{\Gamma}:\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$ which made by vector $\vec{\alpha}$ and $\vec{\beta}$.

Keywords：curves；curvature；torsion；basic vector

已知空间曲线$\Gamma:\vec{r}=\vec{r}(s)$在每一正常点处有切线、主法线、副法线，它们对应的基本向量为单位切向量、主法向量、副法向量，分别记为$\vec{\alpha}(s)$、$\vec{\beta}(s)$、$\vec{\gamma}(s)$，简记为$\vec{\alpha}$、$\vec{\beta}$、$\vec{\gamma}$，且在每一正常点处有曲率$\kappa(s)$和挠率$\tau(s)$，简记为κ、τ，s为自然参数[1-2]。下面研究由已知曲线$\Gamma:\vec{r}=\vec{r}(s)$的基本向量$\vec{\alpha}$、$\vec{\beta}$所作曲线$\bar{\Gamma}:\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$的曲率$\bar{\kappa}$和挠率$\bar{\tau}$的计算问题[3-6]。

命题1　当k，τ都为非0常数时，曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$的曲率和挠率分别为：

$$\bar{\kappa}=\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(a\kappa-\tau)^2+b^2\left(\kappa^2+\tau^2\right)}},\quad \bar{\tau}=0。$$

证明　由曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$，及其Frenet公式得：

$$\vec{\rho}'=\dot{\vec{\gamma}}+a\dot{\vec{\alpha}}+b\dot{\vec{\beta}}=-\tau\vec{\beta}+a\kappa\vec{\beta}+b(-\kappa\vec{\alpha}+\tau\vec{\gamma})=$$
$$-b\kappa\vec{\alpha}+(a\kappa-\tau)\beta+b\tau\vec{\gamma}，$$

$$\vec{\rho}''=-b\kappa\dot{\vec{\alpha}}+(a\kappa-\tau)\dot{\vec{\beta}}+b\tau\dot{\vec{\gamma}}=$$
$$-b\kappa\kappa\vec{\beta}+(a\kappa-\tau)(-\kappa\vec{\alpha}+\tau\vec{\gamma})+b\tau\left(-\tau\vec{\beta}\right)=$$
$$\left(\tau\kappa-a\kappa^2\right)\vec{\alpha}+\left(-b\kappa^2-b\tau^2\right)\vec{\beta}+\left(a\kappa\tau-\tau^2\right)\vec{\gamma},$$

$$\vec{\rho}'''=\left(\tau\kappa-a\kappa^2\right)\dot{\vec{\alpha}}+\left(-b\kappa^2-b\tau^2\right)\dot{\vec{\beta}}+\left(a\kappa\tau-\tau^2\right)\dot{\vec{\gamma}}=$$
$$\left(\tau\kappa-a\kappa^2\right)\kappa\vec{\beta}+\left(-b\kappa^2-b\tau^2\right)\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)+$$
$$\left(a\kappa\tau-\tau^2\right)\left(-\tau\vec{\beta}\right)=$$
$$b\kappa\left(\kappa^2+\tau^2\right)\vec{\alpha}+\left(\tau-a\kappa\right)\left(\kappa^2+\tau^2\right)\vec{\beta}+$$
$$\left(-b\tau\right)\left(\kappa^2+\tau^2\right)\vec{\gamma}.$$

$$\vec{\rho}'^2=\left(a\kappa-\tau^2\right)+b^2\kappa^2+b^2\tau^2=$$
$$\left(a\kappa-\tau^2\right)+b^2\kappa^2+b^2\tau^2,$$

$$\vec{\rho}''^2=\left(\tau\kappa-a\kappa^2\right)^2+\left(a\tau\kappa-\tau^2\right)^2+\left(b\kappa^2+b\tau^2\right)^2，$$

$$\vec{\rho}'\cdot\vec{\rho}''=\left(a\kappa-\tau\right)\left(-b\kappa^2-b\tau^2\right)+$$
$$\left(-b\kappa\right)\left(\tau\kappa-a\kappa^2\right)+\left(b\tau\right)\left(a\kappa\tau-\tau^2\right)=0。$$

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由拉格朗日公式得：

$$\left(\vec{\rho}'\times\vec{\rho}''\right)^2=\begin{vmatrix}\vec{\rho}'^2 & \vec{\rho}'\cdot\vec{\rho}''\\ \vec{\rho}''\cdot\vec{\rho}' & \vec{\rho}''^2\end{vmatrix}=\vec{\rho}'^2\vec{\rho}''^2-\left(\vec{\rho}'\cdot\vec{\rho}''\right)^2=$$

$$\left[\left(a\kappa-\tau^2\right)+b^2\kappa^2+b^2\tau^2\right].$$

$$\left[\left(\tau\kappa-a\kappa^2\right)^2+\left(a\kappa\tau-\tau^2\right)^2+\left(b\kappa^2+b\tau^2\right)^2\right]-0=$$

$$\left(\kappa^2+\tau^2\right)\left[\left(a\kappa-\tau\right)^2+b^2\left(\kappa^2+\tau^2\right)\right]^2,$$

$$\left|\vec{\rho}'\times\vec{\rho}''\right|=\left[\left(a\kappa-\tau^2\right)^2+b^2\left(\kappa^2+\tau^2\right)\right]\sqrt{\kappa^2+\tau^2},$$

代入曲率公式得：

$$\bar{\kappa}=\frac{\left|\vec{\rho}'\times\vec{\rho}''\right|}{\left|\vec{\rho}'\right|^3}=\frac{\left[(a\kappa-\tau)^2+b^2\left(\kappa^2+\tau^2\right)\right]\sqrt{\kappa^2+\tau^2}}{\sqrt{\left[\left(a\kappa-\tau\right)^2+b^2\left(\kappa^2+\tau^2\right)\right]^3}}=$$

$$\sqrt{\frac{\kappa^2+\tau^2}{(a\kappa-\tau)^2+b^2\left(\kappa^2+\tau^2\right)}}。$$

$$\left(\vec{\rho}',\vec{\rho}'',\vec{\rho}'''\right)=$$

$$\begin{vmatrix}-b\kappa & a\kappa-\tau & b\tau\\ \tau\kappa-a\kappa^2 & -b\left(\kappa^2+\tau^2\right) & a\kappa\tau-\tau^2\\ b\kappa\left(\kappa^2+\tau^2\right) & \left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right) & -b\tau\left(\tau^2+\kappa^2\right)\end{vmatrix}=0,$$

代入挠率公式得$\bar{\tau}=\dfrac{\left(\vec{\rho}',\ \vec{\rho}'',\ \vec{\rho}'''\right)}{\left(\vec{\rho}'\times\vec{\rho}''\right)^2}=0$。

推论 当$a=\dfrac{\tau}{\kappa}$时，曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$的曲率$\bar{\kappa}=\pm\dfrac{1}{b}$，因而曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$是1个半径为$\pm b$的圆。

命题2 当κ为非0常数，而τ不为非0常数时，曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$的曲率和挠率分别为：

$$\bar{\kappa}=\sqrt{\frac{\left[b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]\left\{\kappa^2\left(\tau-a\kappa\right)^2+\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2+\left[\tau\left(a\kappa-\tau\right)+b\dot{\tau}\right]^2\right\}-\dot{\tau}^2\left[\left(b^2+1\right)\tau-a\kappa\right]^2}{\left[\left(\tau-a\kappa\right)^2+b^2\left(\kappa^2+\tau^2\right)\right]^3}},$$

$$\bar{\tau}=\frac{\left[-\kappa\tau\dot{\tau}+a\kappa+b\kappa\ddot{\tau}\right](\tau-a\kappa)^2+\left[ab^2\kappa^2+b^3\kappa\ddot{\tau}-b^2\tau\dot{\tau}\kappa\right]\left(\kappa^2+\tau^2\right)+2b\kappa\tau(a\kappa-\tau)+2b\kappa\dot{\tau}^2\tau-3\kappa\tau\dot{\tau}^2-3b^2\kappa\tau\dot{\tau}^2}{\left[b\left(\kappa^2+\tau^2\right)+(a\kappa-\tau)^2\right]\left\{\kappa^2(\tau-a\kappa)^2+\left[\tau(a\kappa-\tau)+b\dot{\tau}\right]^2+\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2\right\}-\dot{\tau}^2\left[\left(b^2+1\right)\tau-a\kappa\right]^2}。$$

证明 由曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$及Frenet公式得：

$$\vec{\rho}'=-\tau\vec{\beta}+a\kappa\vec{\beta}+b\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)=-b\kappa\vec{\alpha}+\left(a\kappa-\tau\right)\vec{\beta}+b\tau\vec{\gamma},$$

$$\vec{\rho}''=-b\kappa\vec{\alpha}'+\left(-\dot{\tau}\right)\vec{\beta}+\left(a\kappa-\tau\right)\vec{\beta}'+b\dot{\tau}\vec{\gamma}+b\tau\vec{\gamma}'=$$

$$-b\kappa\kappa\vec{\beta}-\dot{\tau}\vec{\beta}+\left(a\kappa-\tau\right)\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)++b\dot{\tau}\vec{\gamma}+b\tau\left(-\tau\vec{\beta}\right)=$$

$$\kappa\left(\tau-a\kappa\right)\vec{\alpha}+(-)\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]\vec{\beta}+\left[\tau\left(a\kappa-\tau\right)+b\dot{\tau}\right]\vec{\gamma},$$

$$\vec{\rho}'''=\kappa\dot{\tau}\vec{\alpha}+\kappa\left(\tau-a\kappa\right)\kappa\vec{\beta}+(-)\left(2b\tau\dot{\tau}+\ddot{\tau}\right)\vec{\beta}+$$

$$\left[-b\left(\kappa^2+\tau^2\right)-\dot{\tau}\right]\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)+\left(a\kappa-2\tau\dot{\tau}+b\ddot{\tau}\right)\vec{\gamma}+$$

$$\left[\tau\left(a\kappa-\tau\right)+b\dot{\tau}\right]\left(-\tau\vec{\beta}\right)=\kappa\left[b\left(\kappa^2+\tau^2\right)+2\dot{\tau}\right]\vec{\alpha}+$$

$$\left[\left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-\left(3\tau\dot{\tau}+\ddot{\tau}\right)\right]\vec{\beta}+$$

$$\left[-b\tau\left(\kappa^2+\tau^2\right)-3\tau\dot{\tau}-a\kappa+b\ddot{\tau}\right]\vec{\gamma},$$

$$\vec{\rho}'^2=b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2,$$

$$\vec{\rho}''^2=\kappa^2\left(\tau-a\kappa\right)^2+\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2+\left[\tau\left(a\kappa-\tau\right)+b\dot{\tau}\right]^2,$$

$$\vec{\rho}'\cdot\vec{\rho}''=\dot{\tau}\left[\left(b^2+1\right)\tau-a\kappa\right],$$

$$\left(\vec{\rho}'\times\vec{\rho}''\right)^2=\begin{vmatrix}\vec{\rho}'^2 & \vec{\rho}'\cdot\vec{\rho}''\\ \vec{\rho}''\cdot\vec{\rho}' & \vec{\rho}''^2\end{vmatrix}=\begin{vmatrix}b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2 & \dot{\tau}\left[\left(b^2+1\right)\tau-a\kappa\right]\\ \dot{\tau}\left[\left(b^2+1\right)\tau-a\kappa\right] & \kappa^2\left(\tau-a\kappa\right)^2+\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2+\left[\tau\left(a\kappa-\tau\right)+b\dot{\tau}\right]^2\end{vmatrix}=$$

$$\left[b\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]\left\{\kappa^2\left(\tau-a\kappa\right)^2+\left[\tau\left(a\kappa-\tau\right)+b\dot{\tau}\right]^2+\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2\right\}-\dot{\tau}^2\left[\left(b^2+1\right)\tau-a\kappa\right]^2,$$

代入曲率公式$\bar{\kappa}$得：

$$\bar{\kappa}=\frac{\left|\vec{\rho}'\times\vec{\rho}''\right|}{\left|\vec{\rho}'\right|^3}=\sqrt{\frac{\left[b^2\left(\kappa^2+\tau^2\right)+(a\kappa-\tau)^2\right]\left\{\kappa^2(\tau-a\kappa)^2+\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2+\left[\tau(a\kappa-\tau)+b\dot{\tau}\right]^2\right\}-\dot{\tau}^2\left[\left(b^2+1\right)\tau-a\kappa\right]^2}{\left[(\tau-a\kappa)^2+b^2\left(\kappa^2+\tau^2\right)\right]^3}},$$

$$\left(\vec{\rho}',\ \vec{\rho}'',\ \vec{\rho}'''\right)=\begin{vmatrix}-b\kappa & a\kappa-\tau & b\tau\\ \kappa\left(\tau-a\kappa\right) & -b\left(\kappa^2+\tau^2\right)-\dot{\tau} & \tau\left(a\kappa-\tau\right)+b\dot{\tau}\\ \kappa\left[b\left(\kappa^2+\tau^2\right)+2\dot{\tau}\right] & \left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-\left(3b\tau\dot{\tau}+\ddot{\tau}\right) & -b\tau\left(\kappa^2+\tau^2\right)-3\tau\dot{\tau}+a\kappa+b\ddot{\tau}\end{vmatrix}=$$

$$(-b\kappa)\begin{vmatrix} -\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right] & \tau\left(a\kappa-\tau\right)+b\dot{\tau} \\ \left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-\left(3b\tau\dot{\tau}+\ddot{\tau}\right) & -b\tau\left(\kappa^2+\tau^2\right)-3\tau\dot{\tau}+a\kappa+b\ddot{\tau}\end{vmatrix}+$$

$$(\tau-a\kappa)\begin{vmatrix} \kappa\left(\tau-a\kappa\right) & \tau\left(a\kappa-\tau\right)+b\dot{\tau} \\ \kappa\left[b\left(\kappa^2+\tau^2\right)+2\dot{\tau}\right] & -b\tau\left(\kappa^2+\tau^2\right)-3\tau\dot{\tau}+a\kappa+b\ddot{\tau}\end{vmatrix}+$$

$$b\tau\begin{vmatrix} \kappa\left(\tau-a\kappa\right) & -\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right] \\ \kappa\left[b\left(\kappa^2+\tau^2\right)+2\dot{\tau}\right] & \left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-\left(3b\tau\dot{\tau}+\ddot{\tau}\right)\end{vmatrix}=\left[-\kappa\tau\dot{\tau}+a\kappa+b\kappa\ddot{\tau}\right]\left(\tau-a\kappa\right)^2+$$

$$\left[ab^2\kappa^2+b^3\kappa\ddot{\tau}-b^2\tau\dot{\tau}\kappa\right]\left(\kappa^2+\tau^2\right)+2b\kappa\tau\left(a\kappa-\tau\right)+2b\kappa\dot{\tau}^2\tau-3\kappa\tau\dot{\tau}^2-3b^2\kappa\tau\dot{\tau}^2,$$

代入挠率公式得：

$$\bar{\tau}=\frac{\left(\vec{\rho}',\ \ \vec{\rho}'',\ \ \vec{\rho}'''\right)}{\left(\vec{\rho}'\times\vec{\rho}''\right)^2}=$$

$$\frac{\left[-\kappa\tau\dot{\tau}+a\kappa+b\kappa\ddot{\tau}\right]\left(\tau-a\kappa\right)^2+\left[ab^2\kappa^2+b^3\kappa\ddot{\tau}-b^2\tau\dot{\tau}\kappa\right]\left(\kappa^2+\tau^2\right)+2b\kappa\tau\left(a\kappa-\tau\right)+2b\kappa\dot{\tau}^2\tau-3\kappa\tau\dot{\tau}^2-3b^2\kappa\tau\dot{\tau}^2}{\left[b\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]\left[\kappa^2\left(\tau-a\kappa\right)^2+\left[\tau\left(a\kappa-\tau\right)+b\dot{\tau}\right]^2+\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2\right]-\dot{\tau}^2\left[\left(b^2+1\right)\tau-a\kappa\right]^2}。$$

推论　当$a=\dfrac{\tau}{\kappa}$时，曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$的曲率、挠率分别为：

$$\bar{\kappa}=\sqrt{\frac{\left(\kappa^2+\tau^2\right)\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2+\dot{\tau}^2-b^2\tau^2\dot{\tau}^2}{\left(\kappa^2+\tau^2\right)^3}}、$$

$$\bar{\tau}=\frac{\left(b^2\kappa\tau+b^3\kappa\ddot{\tau}-b^2\tau\dot{\tau}\kappa\right)\left(\kappa^2+\tau^2\right)+\left(2b-3-3b^2\right)\kappa\tau\dot{\tau}^2}{b\left(\kappa^2+\tau^2\right)\left\{b^2\dot{\tau}^2+\left[b\left(\kappa^2+\tau^2\right)+\dot{\tau}\right]^2\right\}-b^4\tau^2\dot{\tau}^2}。$$

命题3　当κ不为非0常数，而τ为非0常数时，曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$的曲率和挠率分别为：

$$\bar{\kappa}=\sqrt{\frac{\left\{\left[\kappa\left(\tau-a\kappa\right)-b\dot{\kappa}\right]^2+\left[a\dot{\kappa}-b\left(\kappa^2+\tau^2\right)\right]^2+\tau^2\left(a\kappa-\tau\right)^2\right\}\left[b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]-\left[\left(a^2+b^2\right)\kappa\dot{\kappa}-a\tau\dot{\kappa}\right]^2}{\left[b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]^3}},$$

$$\bar{\tau}=\frac{\left[\tau\left(\tau-a\kappa\right)^2-ab\tau\dot{\kappa}+b^2\tau\left(\kappa^2+\tau^2\right)\right]\left[\tau\dot{\kappa}-b\ddot{\kappa}-3a\kappa\dot{\kappa}+b\kappa^3+b\kappa\tau^2\right]-b^2\tau\dot{\kappa}\left[\left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-3b\kappa\dot{\kappa}+a\ddot{\kappa}\right]+}{\left\{\left[\kappa\left(\tau-a\kappa\right)-b\dot{\kappa}\right]^2+\left[a\dot{\kappa}-b\left(\kappa^2+\tau^2\right)\right]^2+\tau^2\left(a\kappa-\tau\right)^2\right\}\cdot}\rightarrow$$

$$\leftarrow\frac{\tau\left[2a\dot{\kappa}-b\left(\kappa^2+\tau^2\right)\right]\left[b^2\kappa\left(\kappa^2+\tau^2\right)-ab\kappa\dot{\kappa}+\kappa\left(a\kappa-\tau\right)^2+b\dot{\kappa}\left(a\kappa-\tau\right)\right]}{\left[b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]-\left[\left(a^2+b^2\right)\kappa\dot{\kappa}-a\tau\dot{\kappa}\right]^2}。$$

证明　由曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$及Frenet公式得：

$$\vec{\rho}'=\dot{\vec{\gamma}}+a\dot{\vec{\alpha}}+b\dot{\vec{\beta}}=-\tau\vec{\beta}+a\kappa\vec{\beta}+b\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)=$$
$$-b\kappa\vec{\alpha}+\left(a\kappa-\tau\right)\vec{\beta}+b\tau\vec{\gamma},$$

$$\vec{\rho}''=-b\dot{\kappa}\vec{\alpha}-b\kappa\kappa\vec{\beta}+\left(a\dot{\kappa}\right)\vec{\beta}+\left(a\kappa-\tau\right)\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)+$$
$$b\tau\left(-\tau\vec{\beta}\right)=\left[\kappa\left(\tau-a\kappa\right)-b\dot{\kappa}\right]\vec{\alpha}+$$
$$\left[a\dot{\kappa}-b\left(\kappa^2+\tau^2\right)\right]\vec{\beta}+\tau\left(a\kappa-\tau\right)\vec{\gamma},$$

$$\vec{\rho}'''=\left[\dot{\kappa}\left(\tau-a\kappa\right)+\kappa\left(-a\dot{\kappa}\right)-b\ddot{\kappa}\right]\vec{\alpha}+$$
$$\left[\kappa\left(\tau-a\kappa\right)-b\dot{\kappa}\right]\left(\kappa\vec{\beta}\right)+\left(a\ddot{\kappa}-2b\kappa\dot{\kappa}\right)\vec{\beta}+$$
$$\left[a\dot{\kappa}-b\left(\kappa^2+\tau^2\right)\right]\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)+a\tau\dot{\kappa}\vec{\gamma}+$$
$$\tau\left(a\kappa-\tau\right)\left(-\tau\vec{\beta}\right)=$$
$$\left[\tau\dot{\kappa}-b\ddot{\kappa}-3a\kappa\dot{\kappa}+b\kappa^3+b\kappa\tau^2\right]\vec{\alpha}+$$
$$\left[\left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-3b\kappa\dot{\kappa}+a\ddot{\kappa}\right]\vec{\beta}+$$
$$\tau\left[2a\dot{\kappa}-b\left(\kappa^2+\tau^2\right)\right]\vec{\gamma},$$

$$\vec{\rho}'^2=b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2,$$

$$\vec{\rho}''^2=\left[\kappa\left(\tau-a\kappa\right)-b\dot{\kappa}\right]^2+\left[a\dot{\kappa}-b\left(\kappa^2+\tau^2\right)\right]^2+$$
$$\tau^2\left(a\kappa-\tau\right)^2,$$

$$\vec{\rho}'\cdot\vec{\rho}''=\left(a^2+b^2\right)\kappa\dot{\kappa}-a\tau\dot{\kappa},$$

$$(\vec\rho'\times\vec\rho'')^2=\begin{vmatrix}\vec\rho'^2 & \vec\rho'\cdot\vec\rho'' \\ \vec\rho''\cdot\vec\rho' & \vec\rho''^2\end{vmatrix}=\begin{vmatrix} b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2 & \left(a^2+b^2\right)\kappa\dot\kappa-a\tau\dot\kappa \\ \left(a^2+b^2\right)\kappa\dot\kappa-a\tau\dot\kappa & \left[\kappa\left(\tau-a\kappa\right)-b\dot\kappa\right]^2+\left[a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right]^2+\tau^2\left(a\kappa-\tau\right)^2\end{vmatrix}=$$

$$\left[b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]\left\{\left[\kappa\left(\tau-a\kappa\right)-b\dot\kappa\right]^2+\left[a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right]^2+\tau^2\left(a\kappa-\tau\right)^2\right\}-\left[\left(a^2+b^2\right)\kappa\dot\kappa-a\tau\dot\kappa\right]^2,$$

代入曲率公式得：

$$\bar\kappa=\frac{\left|\vec\rho'\times\vec\rho''\right|}{\left|\vec\rho'\right|^3}=$$

$$\sqrt{\frac{\left\{\left[\kappa\left(\tau-a\kappa\right)-b\dot\kappa\right]^2+\left[a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right]^2+\tau^2\left(a\kappa-\tau\right)^2\right\}\left[b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]-\left[\left(a^2+b^2\right)\kappa\dot\kappa-a\tau\dot\kappa\right]^2}{\left[b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]^3}},$$

$$(\vec\rho',\ \vec\rho'',\ \vec\rho''')=\begin{vmatrix} -b\kappa & a\kappa-\tau & b\tau \\ \kappa\left(\tau-a\kappa\right)-b\dot\kappa & a\dot\kappa-b\left(\kappa^2+\tau^2\right) & \tau\left(a\kappa-\tau\right) \\ \tau\dot\kappa-b\ddot\kappa-3a\kappa\dot\kappa+b\kappa^3+b\kappa\tau^2 & \left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-3b\kappa\dot\kappa+a\ddot\kappa & \tau\left[2a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right]\end{vmatrix}=$$

$$\left[\tau\dot\kappa-b\ddot\kappa-3a\kappa\dot\kappa+b\kappa^3+b\kappa\tau^2\right]\begin{vmatrix} a\kappa-\tau & b\tau \\ a\dot\kappa-b\left(\kappa^2+\tau^2\right) & \tau\left(a\kappa-\tau\right)\end{vmatrix}+(-)\left[\left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-3b\kappa\dot\kappa+a\ddot\kappa\right].$$

$$\begin{vmatrix} -b\kappa & b\tau \\ \kappa\left(\tau-a\kappa\right)-b\dot\kappa & \tau\left(a\kappa-\tau\right)\end{vmatrix}+\tau\left[2a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right]\begin{vmatrix} -b\kappa & a\kappa-\tau \\ \kappa\left(\tau-a\kappa\right)-b\dot\kappa & \tau\left(a\kappa-\tau\right)\end{vmatrix}=$$

$$\left[\tau\dot\kappa-b\ddot\kappa-3a\kappa\dot\kappa+b\kappa^3+b\kappa\tau^2\right]\left[\tau\left(\tau-a\kappa\right)^2-ab\tau\dot\kappa+b^2\tau\left(\kappa^2+\tau^2\right)\right]+$$
$$\left[\left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-3b\kappa\dot\kappa+a\ddot\kappa\right]\left[b\kappa\tau\left(a\kappa-\tau\right)-b\kappa\tau\left(a\kappa-\tau\right)-b^2\tau\dot\kappa\right]+\tau\left[2a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right].$$
$$\left[b^2\kappa\left(\kappa^2+\tau^2\right)-ab\kappa\dot\kappa+\kappa\left(a\kappa-\tau\right)^2+b\dot\kappa\left(a\kappa-\tau\right)\right]=$$
$$\left[\tau\dot\kappa-b\ddot\kappa-3a\kappa\dot\kappa+b\kappa^3+b\kappa\tau^2\right]\left[\tau\left(\tau-a\kappa\right)^2-ab\tau\dot\kappa+b^2\tau\left(\kappa^2+\tau^2\right)\right]+\left(-b^2\tau\dot\kappa\right)\left[\left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-3b\kappa\dot\kappa+a\ddot\kappa\right]+$$
$$\tau\left[2a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right]\left[b^2\kappa\left(\kappa^2+\tau^2\right)-ab\kappa\dot\kappa+\kappa\left(a\kappa-\tau\right)^2+b\dot\kappa\left(a\kappa-\tau\right)\right],$$

代入挠率公式得：

$$\bar\tau=\frac{(\vec\rho',\ \vec\rho'',\ \vec\rho''')}{(\vec\rho'\times\vec\rho'')^2}=$$

$$\frac{\left[\tau\dot\kappa-b\ddot\kappa-3a\kappa\dot\kappa+b\kappa^3+b\kappa\tau^2\right]\left[\tau\left(\tau-a\kappa\right)^2-ab\tau\dot\kappa+b^2\tau\left(\kappa^2+\tau^2\right)\right]+\left(-b^2\tau\dot\kappa\right)\left[\left(\kappa^2+\tau^2\right)\left(\tau-a\kappa\right)-3b\kappa\dot\kappa+a\ddot\kappa\right]+}{\left[b^2\left(\kappa^2+\tau^2\right)+\left(a\kappa-\tau\right)^2\right]\left[\left[\kappa\left(\tau-a\kappa\right)-b\dot\kappa\right]^2+\left[a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right]^2+\tau^2\left(a\kappa-\tau\right)^2\right]-}\rightarrow$$

$$\leftarrow\frac{\tau\left[2a\dot\kappa-b\left(\kappa^2+\tau^2\right)\right]\left[b^2\kappa\left(\kappa^2+\tau^2\right)-ab\kappa\dot\kappa+\kappa\left(a\kappa-\tau\right)^2+b\dot\kappa\left(a\kappa-\tau\right)\right]}{\left[\left(a^2+b^2\right)\kappa\dot\kappa-a\tau\dot\kappa\right]^2}。$$

命题 4 当κ，τ都不为非0常数时，曲线 $\vec\rho=\vec r+a\vec\alpha+b\vec\beta$的曲率和挠率分别为：

$$\bar\kappa=\left\{\left\{\left[\tau^2\kappa^2+\left(\kappa^4+\dot\kappa^2\right)\left(a^2+b^2\right)+\tau^4\left(b^2+1+a^2\kappa^2\right)+\dot\tau^2\left(b^2+1\right)-2a\kappa\tau\left(\kappa^2+\tau^2\right)-2b\tau\kappa\dot\kappa-2a\dot\kappa\dot\tau-2ab\kappa\tau^2+\right.\right.\right.$$
$$2b\dot\tau\kappa^2+2ab\kappa\tau\dot\tau+2b^2\kappa^2\tau^2\Big]\left[b\left(\kappa^2+\tau^2\right)+a^2\kappa^2+\tau^2-2a\kappa\tau\right]-$$
$$\left.\left.\left[\kappa\dot\kappa\left(a^2+b^2\right)+\tau\dot\tau\left(b^2+1\right)-a\left(\kappa\dot\tau+\dot\kappa\tau\right)\right]^2\right\}\left[b\left(\kappa^2+\tau^2\right)+a^2\kappa^2+\tau^2-2a\kappa\tau\right]^{(-3)}\right\}^{\frac{1}{2}},$$

$$\bar{\tau}=\frac{\left[\kappa\dot{\tau}+\tau\dot{\kappa}+\kappa\tau-3a\kappa\dot{\kappa}-b\ddot{\kappa}+b\kappa^{3}+b\kappa\tau^{2}\right]\left[ab\kappa\dot{\tau}-a^{2}\kappa^{2}\tau^{2}+a\kappa\tau^{2}+\tau^{3}-ab\tau\dot{\kappa}+b^{2}\tau\kappa^{2}+b^{2}\tau^{3}\right]+}{[\tau^{2}\kappa^{2}+\ \ \kappa^{4}+\dot{\kappa}^{2}\ \left(a^{2}+b^{2}\right)+\tau^{4}\left(b^{2}+1+a^{2}\kappa^{2}\right)+\dot{\tau}^{2}\left(b^{2}+1\right)-2a\kappa\tau\left(\kappa^{2}+\tau^{2}\right)-2b\tau\kappa\dot{\kappa}-2a\dot{\kappa}\dot{\tau}-2ab\kappa\tau^{2}+}\rightarrow$$

$$\leftarrow\frac{\left[\tau\kappa^{2}-a\kappa^{3}-3b\kappa\dot{\kappa}+a\ddot{\kappa}-\ddot{\tau}-3b\tau\ddot{\tau}-a\kappa\tau^{2}+\tau^{3}\right].\left[b^{2}\left(\kappa\dot{\tau}-\tau\dot{\kappa}\right)\right]+}{2b\dot{\tau}\kappa^{2}+2ab\kappa\tau\dot{\tau}+2b^{2}\kappa^{2}\tau^{2}]\left[b\left(\kappa^{2}+\tau^{2}\right)+a^{2}\kappa^{2}+\tau^{2}-2a\kappa\tau\right]-}\rightarrow$$

$$\leftarrow\frac{\left[2a\tau\dot{\kappa}-\tau^{2}-b\tau\kappa^{2}-b\tau^{3}+b\ddot{\tau}+a\kappa\dot{\tau}-2\tau\dot{\tau}\right]\left[\kappa^{3}\left(a^{2}+b^{2}\right)+b\left(\kappa\dot{\tau}-\tau\dot{\kappa}\right)+\kappa\tau^{2}\left(b^{2}+1\right)-2a\tau\kappa^{2}\right]}{\left[\kappa\dot{\kappa}\left(a^{2}+b^{2}\right)+\tau\dot{\tau}\left(b^{2}+1\right)-a\left(\kappa\dot{\tau}+\dot{\kappa}\tau\right)\right]^{2}}。$$

证明　由曲线$\vec{\rho}=\vec{r}+a\vec{\alpha}+b\vec{\beta}$及Frenet公式得

$$\vec{\rho}'=\dot{\vec{\gamma}}+a\dot{\vec{\alpha}}+b\dot{\vec{\beta}}=-\tau\vec{\beta}+a\kappa\vec{\beta}+b\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)=$$
$$-b\kappa\vec{\alpha}+\left(a\kappa-\tau\right)\vec{\beta}+b\tau\vec{\gamma},$$

$$\vec{\rho}''=-b\kappa\vec{\alpha}-b\kappa\kappa\vec{\beta}+\left(a\dot{\kappa}-\dot{\tau}\right)\vec{\beta}+\left(a\kappa-\tau\right)\left(-\kappa\vec{\alpha}+\tau\vec{\gamma}\right)+$$
$$b\dot{\tau}\vec{\gamma}+b\tau\left(-\tau\vec{\beta}\right)=$$
$$\left(\tau\kappa-a\kappa^{2}-b\dot{\kappa}\right)\vec{\alpha}+\left[a\dot{\kappa}-\dot{\tau}-b\left(\kappa^{2}+\tau^{2}\right)\right]\vec{\beta}+$$
$$\left(b\dot{\tau}+a\kappa\tau-\tau^{2}\right)\vec{\gamma}\ ,$$

$$\vec{\rho}'''=\left(\tau\dot{\kappa}+\kappa\dot{\tau}-2a\kappa\dot{\kappa}-2b\tau\dot{\tau}\right)\vec{\alpha}+\left(\tau\kappa-a\kappa^{2}-b\dot{\kappa}\right)\kappa\vec{\beta}+$$
$$\left(a\ddot{\kappa}-\ddot{\tau}-2b\kappa\dot{\kappa}-2b\tau\dot{\tau}\right)\vec{\beta}+\left(a\dot{\kappa}-\dot{\tau}-b\left(\kappa^{2}+\tau^{2}\right)\right)\left(-\kappa\vec{\alpha}+\right.$$
$$\left.\tau\vec{\gamma}\right)+\left(b\ddot{\tau}+a\dot{\kappa}\tau+a\dot{\tau}\kappa-2\tau\dot{\tau}\right)\vec{\gamma}+\left(b\dot{\tau}+a\kappa\tau-\tau^{2}\right)\left(-\tau\vec{\beta}\right)=$$
$$\left(\kappa\dot{\tau}+\tau\dot{\kappa}+\kappa\tau-3a\kappa\dot{\kappa}-b\ddot{\kappa}+b\kappa^{3}+b\kappa\tau^{2}\right)\vec{\alpha}+$$
$$\left(\tau\kappa^{2}-a\kappa^{3}-3b\kappa\dot{\kappa}+a\ddot{\kappa}-\ddot{\tau}-3b\tau\ddot{\tau}-a\kappa\tau^{2}+\tau^{3}\right)\vec{\beta}+$$
$$\left(2a\tau\dot{\kappa}-\tau^{2}-b\tau\kappa^{2}-b\tau^{3}+b\ddot{\tau}+a\kappa\dot{\tau}-2\tau\dot{\tau}\right)\vec{\gamma}\ ,$$

$$\vec{\rho}'^{2}=b^{2}\left(\kappa^{2}+\tau^{2}\right)+\left(a\kappa-\tau\right)^{2},$$

$$\vec{\rho}''^{2}=\left(\tau\kappa-a\kappa^{2}-b\dot{\kappa}\right)^{2}+\left(a\dot{\kappa}-\dot{\tau}-b\left(\kappa^{2}+\tau^{2}\right)\right)^{2}+$$
$$\left(b\dot{\tau}+a\kappa\tau-\tau^{2}\right)^{2}=\left[\tau^{2}\kappa^{2}+\ \left(\kappa^{4}+\dot{\kappa}^{2}\right)\left(a^{2}+b^{2}\right)+\right.$$
$$\tau^{4}\left(b^{2}+1+a^{2}\kappa^{2}\right)+\dot{\tau}^{2}\left(b^{2}+1\right)-2a\kappa\tau\left(\kappa^{2}+\tau^{2}\right)-$$
$$\left.2b\tau\kappa\dot{\kappa}-2a\dot{\kappa}\dot{\tau}-2ab\kappa\tau^{2}+2b\dot{\tau}\kappa^{2}+2ab\kappa\tau\dot{\tau}+2b^{2}\kappa^{2}\tau^{2}\right],$$

$$\vec{\rho}'\cdot\vec{\rho}''=\kappa\dot{\kappa}\left(a^{2}+b^{2}\right)+\tau\dot{\tau}\left(b^{2}+1\right)-a\left(\kappa\dot{\tau}+\dot{\kappa}\tau\right),$$

$$\left(\vec{\rho}'\times\vec{\rho}''\right)^{2}=\begin{vmatrix}\vec{\rho}'^{2} & \vec{\rho}'\cdot\vec{\rho}''\\ \vec{\rho}''\cdot\vec{\rho}' & \vec{\rho}''^{2}\end{vmatrix}=$$

$$\begin{vmatrix}b^{2}\left(\kappa^{2}+\tau^{2}\right)+\left(a\kappa-\tau\right)^{2} & \kappa\dot{\kappa}\left(a^{2}+b^{2}\right)+\tau\dot{\tau}\left(b^{2}+1\right)-a\left(\kappa\dot{\tau}+\dot{\kappa}\tau\right)\\ \kappa\dot{\kappa}\left(a^{2}+b^{2}\right)+\tau\dot{\tau}\left(b^{2}+1\right)-a\left(\kappa\dot{\tau}+\dot{\kappa}\tau\right) & \left(\tau\kappa-a\kappa^{2}-b\dot{\kappa}\right)^{2}+\left(a\dot{\kappa}-\dot{\tau}-b\left(\kappa^{2}+\tau^{2}\right)\right)^{2}+\left(b\dot{\tau}+a\kappa\tau-\tau^{2}\right)^{2}\end{vmatrix}=$$

$$\left[\tau^{2}\kappa^{2}+\left(\kappa^{4}+\dot{\kappa}^{2}\right)\left(a^{2}+b^{2}\right)+\tau^{4}\left(b^{2}+1+a^{2}\kappa^{2}\right)+\dot{\tau}^{2}\left(b^{2}+1\right)-2a\kappa\tau\left(\kappa^{2}+\tau^{2}\right)-2b\tau\kappa\dot{\kappa}-2a\dot{\kappa}\dot{\tau}-2ab\kappa\tau^{2}+\right.$$
$$\left.2b\dot{\tau}\kappa^{2}+2ab\kappa\tau\dot{\tau}+2b^{2}\kappa^{2}\tau^{2}\right]\left[b^{2}\left(\kappa^{2}+\tau^{2}\right)+\left(a\kappa-\tau\right)^{2}\right]-\left[\kappa\dot{\kappa}\left(a^{2}+b^{2}\right)+\tau\dot{\tau}\left(b^{2}+1\right)-a\left(\kappa\dot{\tau}+\dot{\kappa}\tau\right)\right]^{2}。$$

$$\left(\vec{\rho}',\ \vec{\rho}'',\ \vec{\rho}'''\right)=$$

$$\begin{vmatrix}-b\kappa & a\kappa-\tau & b\tau\\ \tau\kappa-a\kappa^{2}-b\dot{\kappa} & a\dot{\kappa}-\dot{\tau}-b\left(\kappa^{2}+\tau^{2}\right) & b\dot{\tau}+a\kappa\tau-\tau^{2}\\ \kappa\dot{\tau}+\tau\dot{\kappa}+\kappa\tau-3a\kappa\dot{\kappa}-b\ddot{\kappa}+b\kappa^{3}+b\kappa\tau^{2} & \tau\kappa^{2}-a\kappa^{3}-3b\kappa\dot{\kappa}+a\ddot{\kappa}-\ddot{\tau}-3b\tau\ddot{\tau}-a\kappa\tau^{2}+\tau^{3} & 2a\tau\dot{\kappa}-\tau^{2}-b\tau\kappa^{2}-b\tau^{3}+b\ddot{\tau}+a\kappa\dot{\tau}-2\tau\dot{\tau}\end{vmatrix}=$$

$$\left[\kappa\dot{\tau}+\tau\dot{\kappa}+\kappa\tau-3a\kappa\dot{\kappa}-b\ddot{\kappa}+b\kappa^{3}+b\kappa\tau^{2}\right]\begin{vmatrix}a\kappa-\tau & b\tau\\ a\dot{\kappa}-\dot{\tau}-b\left(\kappa^{2}+\tau^{2}\right) & b\dot{\tau}+a\kappa\tau-\tau^{2}\end{vmatrix}+$$

$$(-)\left[\tau\kappa^{2}-a\kappa^{3}-3b\kappa\dot{\kappa}+a\ddot{\kappa}-\ddot{\tau}-3b\tau\ddot{\tau}-a\kappa\tau^{2}+\tau^{3}\right]\begin{vmatrix}-b\kappa & b\tau\\ \tau\kappa-a\kappa^{2}-b\dot{\kappa} & b\dot{\tau}+a\kappa\tau-\tau^{2}\end{vmatrix}+$$

$$\left[2a\tau\dot{\kappa}-\tau^{2}-b\tau\kappa^{2}-b\tau^{3}+b\ddot{\tau}+a\kappa\dot{\tau}-2\tau\dot{\tau}\right]\begin{vmatrix}-b\kappa & a\kappa-\tau\\ \tau\kappa-a\kappa^{2}-b\dot{\kappa} & a\dot{\kappa}-\dot{\tau}-b\left(\kappa^{2}+\tau^{2}\right)\end{vmatrix}=$$

$$\left[\kappa\dot{\tau}+\tau\dot{\kappa}+\kappa\tau-3a\kappa\dot{\kappa}-b\ddot{\kappa}+b\kappa^{3}+b\kappa\tau^{2}\right]\left[ab\kappa\dot{\tau}-a^{2}\kappa^{2}\tau^{2}+a\kappa\tau^{2}+\tau^{3}-ab\tau\dot{\kappa}+b^{2}\tau\kappa^{2}+b^{2}\tau^{3}\right]+$$
$$\left[\tau\kappa^{2}-a\kappa^{3}-3b\kappa\dot{\kappa}+a\ddot{\kappa}-\ddot{\tau}-3b\tau\ddot{\tau}-a\kappa\tau^{2}+\tau^{3}\right]\left[b^{2}\left(\kappa\dot{\tau}-\tau\dot{\kappa}\right)\right]+$$
$$\left[2a\tau\dot{\kappa}-\tau^{2}-b\tau\kappa^{2}-b\tau^{3}+b\ddot{\tau}+a\kappa\dot{\tau}-2\tau\dot{\tau}\right]\left[\kappa^{3}\left(a^{2}+b^{2}\right)+b\left(\kappa\dot{\tau}-\tau\dot{\kappa}\right)+\kappa\tau^{2}\left(b^{2}+1\right)-2a\tau\kappa^{2}\right],$$

代入曲率及挠率公式得：

$$\bar{\kappa}=\frac{\left|\vec{\rho}'\times\vec{\rho}''\right|}{\left|\vec{\rho}'\right|^{3}}=\Big\{\Big\{\Big[\tau^2\kappa^2+\left(\kappa^4+\dot{\kappa}^2\right)\left(a^2+b^2\right)+\tau^4\left(b^2+1+a^2\kappa^2\right)+\dot{\tau}^2\left(b^2+1\right)-2a\kappa\tau\left(\kappa^2+\tau^2\right)-2b\tau\kappa\dot{\kappa}-$$

$$2a\dot{\kappa}\dot{\tau}-2ab\kappa\tau^2+2b\dot{\tau}\kappa^2+2ab\kappa\tau\dot{\tau}+2b^2\kappa^2\tau^2\Big]\Big[b\left(\kappa^2+\tau^2\right)+a^2\kappa^2+\tau^2-2a\kappa\tau\Big]-$$

$$\Big[\kappa\dot{\kappa}\left(a^2+b^2\right)+\tau\dot{\tau}\left(b^2+1\right)-a\left(\kappa\dot{\tau}+\dot{\kappa}\tau\right)\Big]^2\Big\}\Big[b\left(\kappa^2+\tau^2\right)+\mathrm{a}^2\kappa^2+\tau^2-2a\kappa\tau\Big]^{(-3)}\Big\}^{\frac{1}{2}},$$

$$\bar{\tau}=\frac{\left(\vec{\rho}',\quad\vec{\rho}'',\quad\vec{\rho}'''\right)}{\left|\vec{\rho}'\times\vec{\rho}''\right|^{2}}=$$

$$\frac{\left[\kappa\dot{\tau}+\tau\dot{\kappa}+\kappa\tau-3a\kappa\dot{\kappa}-b\ddot{\kappa}+b\kappa^3+b\kappa\tau^2\right]\left[ab\kappa\dot{\tau}-a^2\kappa^2\tau^2+a\kappa\tau^2+\tau^3-ab\tau\dot{\kappa}+b^2\tau\kappa^2+b^2\tau^3\right]+}{\left[\tau^2\kappa^2+\quad\kappa^4+\dot{\kappa}^2\ \ \left(a^2+b^2\right)+\tau^4\left(b^2+1+a^2\kappa^2\right)+\dot{\tau}^2\left(b^2+1\right)\right.}\rightarrow$$

$$\leftarrow\frac{\left[\tau\kappa^2-a\kappa^3-3b\kappa\dot{\kappa}+a\ddot{\kappa}-\ddot{\tau}-3b\tau\ddot{\tau}-a\kappa\tau^2+\tau^3\right]\left[b^2\left(\kappa\dot{\tau}-\tau\dot{\kappa}\right)\right]\cdot}{\left.(-2)a\kappa\tau\left(\kappa^2+\tau^2\right)-2b\tau\kappa\dot{\kappa}-2a\dot{\kappa}\dot{\tau}-2ab\kappa\tau^2+2b\dot{\tau}\kappa^2+2ab\kappa\tau\dot{\tau}+2b^2\kappa^2\tau^2\right]\cdot}\rightarrow$$

$$\leftarrow\frac{\left[2a\tau\dot{\kappa}-\tau^2-b\tau\kappa^2-b\tau^3+b\ddot{\tau}+a\kappa\dot{\tau}-2\tau\dot{\tau}\right]\left[\kappa^3\left(a^2+b^2\right)+b\left(\kappa\dot{\tau}-\tau\dot{\kappa}\right)+\kappa\tau^2\left(b^2+1\right)-2a\tau\kappa^2\right]}{\left[b\left(\kappa^2+\tau^2\right)+a^2\kappa^2+\tau^2-2a\kappa\tau\right]-\left[\kappa\dot{\kappa}\left(a^2+b^2\right)+\tau\dot{\tau}\left(b^2+1\right)-a\left(\kappa\dot{\tau}+\dot{\kappa}\tau\right)\right]^2}。$$

参考文献：

[1] 刘学泳. 空间两曲线的基本向量之间的关系研究[J]. 湘潭师范学院学报：自然科学版，2004，26(1)：34-38.
Liu Xueyong. On the Relationship between the Fundamental Vectors of Two Curves in Space[J]. Journal of Xiangtan Normal University：Natural Science Edition，2004，26(1)：34-38.

[2] 刘学泳，滕　超，肖前军. 曲线的基本向量所作曲线的曲率与挠率的计算[J]. 湘潭师范学院学报：自然科学版，2005，27(4)：17-21.
Liu Xueyong，Teng Chao，Xiao Qianjun. Calculation of the Curvature and Torsion of Curves Which are Made up Basic Vector[J]. Journal of Xiangtan Normal University：Natural Science Edition，2005，27(4)：17-21.

[3] Hicks N J. Notes on Differential Geometry[M]. Princeton：Princeton University Press，1965.

[4] Kobayashi S. Nomizu K. Foundations of Differential Geometry (Vol. Ⅰ)[M]. New York：Interscience Publishers，1963.

[5] Kobayashi S. Nomizu K. Foundations of Differential Geometry (Vol. Ⅱ)[M]. New York：Interscience Publishers，1969.

[6] Klingenberg W. Riemannian Geometry[M]. New York：Springer-Verlag，1982.

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