矩阵方程 (A^*XA， B^*XB) =(C，D)有Hermite部分是半正定的解与Hermite半正定解的条件

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摘　要：研究了复矩阵方程(A^*XA, B^*XB)=(C, D)有Hermite部分是半正定的解与Hermite半正定解的可解性条件。利用广义奇异值分解，导出了矩阵方程(A^*XA, B^*XB)=(C, D)有Hermite部分是半正定的解、Hermite半正定的解的充分必要条件，同时给出了解的通式。

关键词：矩阵方程；Hermite半正定；广义奇异值分解

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The Solvability Conditions with Hermite Part Positive Semidefinite and Hermite Positive Semidefinite Solutions of the Matrix Equation (A^*XA, B^*XB)=(C, D)

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Abstract：The complex matrix solutions with Hermite part positive semidefinite and Hermite positive semidefinite for the matrix equation (A^*XA, B^*XB)=(C, D) is investigated. The necessary and sufficient conditions for the solvability of such solutions are derived by using the generalized singular value decomposition. The forms of general solution are also given out.

Key words：matrix equation；Hermite positive semidefinite；generalized singular value decomposition

1 Introduction

Let $\boldsymbol{C}^{m\times n}$ denote the set of all $m\times n$ complex matrices，$\boldsymbol{UC}^{n\times n}$ the set of all unitary matrices in $\boldsymbol{C}^{n\times n}$，$\boldsymbol{HC}^{n\times n}$ the set of all Hermite matrices in $\boldsymbol{C}^{n\times n}$，$\boldsymbol{HC}_0^{n\times n}$ the set of all Hermite positive semidefinite matrices in $\boldsymbol{C}^{n\times n}$，$\boldsymbol{C}_0^{n\times n}$ the set of all $n\times n$ complex matrices with Hermite part positive semidefinite，i.e., $\boldsymbol{C}_0^{n\times n}=\{A\in \boldsymbol{C}^{n\times n} \mid A+A^*\in \boldsymbol{HC}_0^{n\times n}\}$，and $\boldsymbol{W}^{r\times t}=\{A\in \boldsymbol{C}^{r\times t}\ \sigma(A)\le 1\}$，where $\sigma(A)$ denotes the set of singular values of matrix A. One element in this set is usually called a contraction. The symbols $A^{\mathrm{T}}, A^{+}, A^{*}$ will stand for the transpose，the Moore-Penrose inverse，conjugate transpose，respectively，of $A\in \boldsymbol{C}^{n\times n}$. $A^{\frac{1}{2}}$ denotes the square root of a Hermite positive semidefinite matrix A. $A\ge 0$ means that A is a complex matrix with Hermite part positive semidefinite.

Dai Hua and Peter Lancaster[1] considered the linear matrix equation $A^{\mathrm{T}}XA=C$ and ($A^{\mathrm{T}}XA$，$XA-YAD$)=(D, 0)，which is motivated and illustrated with an inverse problem of vibration theory. Solvability conditions for symmetric positive semidefinite solution and general solution were obtained by using the singular value decomposition. Liao Anping[2] considered more general matrix equation ($A^{\mathrm{T}}XA$, $B^{\mathrm{T}}XB$)=(C, D) and gave a necessary and sufficient condition for the existence of symmetric positive semidefinite solution，as well as an expression for the general solution，using the generalized singular value decomposition. These linear matrix equations have been studied by many authors see

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references [3,4] and their references.

In this paper, we consider complex positive semidefinite solution of the following matrix equation

$$\begin{cases} A^* XA = C, \\ B^* XB = D, \end{cases} \tag{1}$$

where $A \in C^{m\times n}$, $B \in C^{m\times p}$, $C \in C^{n\times n}$, $D \in C^{p\times p}$. In section 2, we consider the solution with positive semidefinite Hermite part. The necessary and sufficient conditions having such solution are derived, and four expressions for the general solution are given using the generalized singular value decomposition. In particular, when B=0 and D=0, these results mean that we have given the general forms of the solution with positive semidefinite Hermite part for the equation $A^*XA=C$. In Section 3, we investigate the solution of Hermite positive semidefinite for matrix equation (1). The equivalent conditions for the solvability and expressions for the general solution are given.

2 The solution with Hermite part positive semidefinite for the matrix equation $\left(A^* XA, B^* XB\right) = \left(C, D\right)$

The following lemma can be found in reference[5, lemma 3.5.12].

Lemma 1 Let $L \in HC^{m\times m}$, $M \in HC^{n\times n}$ and $X \in C^{m\times n}$ be given. Then $\begin{bmatrix} L & X \\ X^* & M \end{bmatrix} \in C^{(m+n)\times(m+n)}$ is Hermite positive semidefinited, if and only if L and M are Hermite positive semidefinited, and there is a contraction $\omega \in W^{(m\times n)}$, such that $X = L^{\frac{1}{2}} \omega M^{\frac{1}{2}}$.

Lemma 2 Let $A_{X_{13}} \in C^{n\times n}$ have the form

$$A_{X_{13}} = \begin{bmatrix} A_{11} & A_{12} & X_{13} \\ A_{21} & A_{22} & A_{23} \\ X_{31} & A_{32} & A_{33} \end{bmatrix},$$

where $A_{11} \in C^{r\times r}$, $A_{22} \in C^{S\times S}$, all A_{ij} are given and X_{13}, X_{31} are arbitrary. Then $A_{X_{13}} \ge 0$ if and only if

i) $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \ge 0$, $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \ge 0$,

ii) $X_{13} = B_{12}B_{22}^{+}B_{23} + [B_{11} - B_{12}B_{22}^{-}B_{12}^{*}]^{\frac{1}{2}}\omega[B_{33} - B_{23}^{*}B_{22}^{-}B_{23}]^{\frac{1}{2}} - X_{31}^{*}$,

where for $1 \le i, j \le 3$, $(i,j) \ne (i,3)$, $B_{ij} = A_{ij} + A_{ji}^{*}$, $\omega \in W^{r\times(n-r-s)}$ and X_{31} are arbitrary.

Proof "⇒" If $A_{X_{13}} \ge 0$, then it is clear that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \ge 0, \quad \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \ge 0.$$

Put

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}.$$

Then we have that $A + A^* = \begin{bmatrix} A_{11} + A_{11}^* & A_{12} + A_{21}^* \\ A_{21} + A_{12}^* & A_{22} + A_{22}^* \end{bmatrix}$,

$B + B^* = \begin{bmatrix} A_{22} + A_{22}^* & A_{23} + A_{32}^* \\ A_{32} + A_{23}^* & A_{33} + A_{33}^* \end{bmatrix}$ are Hermite positive semidefinite.

It follows from Lemma 1 in reference [7], that

$$A_{12} + A_{21}^* = (A_{12} + A_{21}^*)(A_{22} + A_{22}^*)^{+}(A_{22} + A_{22}^*),$$
$$A_{23} + A_{32}^* = (A_{22} + A_{22}^*)(A_{22} + A_{22}^*)^{+}(A_{23} + A_{32}^*).$$

Put

$$A_{X_{13}} + A_{X_{13}}^* = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix},$$

where $B_{ij} = A_{ij} + A_{ji}^*$, $1 \le i, j \le 3$, and $A_{13} = X_{13}$, $A_{31} = X_{31}$.

$$\text{Set } p = \begin{bmatrix} 0 & I_{r\times r} & 0 \\ I_{s\times s} & -B_{22}^{-}B_{21} & -B_{22}^{-}B_{23} \\ 0 & 0 & I_{(n-r-s)\times(n-r-s)} \end{bmatrix}.$$

Then P is a nonsingular matrix in $C^{n\times n}$ and

$$p^*(A_{X_{13}} + A_{X_{13}}^*)p = \begin{bmatrix} B_{22} & 0 & 0 \\ 0 & B_{11} - B_{12}B_{22}^{+}B_{12}^{*} & B_{13} - B_{12}B_{22}^{+}B_{23} \\ 0 & B_{13}^{*} - B_{23}^{*}B_{22}^{-}B_{12}^{*} & B_{33} - B_{23}^{*}B_{22}^{+}B_{23} \end{bmatrix} = \begin{bmatrix} B_{22} & 0 \\ 0 & \bar{B} \end{bmatrix}.$$

Since $A_{X_{13}}$ is a complex positive semidefinite matrix, we have that $P^*(A_{X_{13}} + A_{X_{13}}^*)P$ is a Hermite positive semidefinite. So $\bar{B}$ is a Hermite positive semidefinite. Thus, by Lemma 1, there exists a contraction $\omega \in C^{r\times(n-r-s)}$, such that

$$B_{13} - B_{12}B_{22}^{+}B_{23} = (B_{11} - B_{12}B_{22}^{+}B_{12}^{*})^{\frac{1}{2}}\omega(B_{33} - B_{23}^{*}B_{22}^{-}B_{23})^{\frac{1}{2}}.$$

It follows that X_{13} has the desired general form. So ii) holds.

"⇐" If i) and ii) hold, it is evident that

$$B_{22} \ge 0, \quad B_{11} - B_{12}B_{22}^{+}B_{12}^{*} \ge 0, \; B_{33} - B_{23}^{*}B_{22}B_{23} \ge 0,$$

and $B_{12}=B_{12}B_{22}^{+}B_{22}$，$B_{23}=B_{22}B_{22}^{+}B_{23}$.

So $P^*(A_{X_{12}}+A_{X_{13}}^*)P=\begin{bmatrix} B_{22} & 0 \\ 0 & \bar{B} \end{bmatrix}$，where P and $\bar{B}$ are as above.

Moreover，it follows from ii) and lemma 1 that $\bar{B}\geq 0$.

Thus $\begin{bmatrix} B_{22} & 0 \\ 0 & \bar{B} \end{bmatrix}\geq 0$. That is $A_{X_{13}}+A_{X_{13}}^*\geq 0$ and $A_{X_{13}}\geq 0$.

The following result is from the Lemma 2 in reference [6].

Lemma 3 Let $L\in HC^{m\times m}$，$M\in HC^{n\times n}$，$X\in C^{m\times n}$ be given. Then $\begin{bmatrix} L & X \\ X^* & M \end{bmatrix}\in C^{(m+n)\times(m+n)}$ is Hermite positive semedefinite，if and only if L is Hermite positive semedefinite and there exists $K\in C^{m\times n}$ such that $X=LK$，and $M-K^*LK$ is Hermite positive semidefinite.

Lemma 4 Let $A_{X,Y}\in C^{n\times n}$ have the form

$$A_{X,Y}=\begin{bmatrix} A_{11} & Y \\ A_{21} & X \end{bmatrix},$$

where $A_{11}\in C^{r\times r}$, $A_{21}\in C^{(n-r)\times r}$ are given and $X\in C^{(n-r)\times(n-r)}$, $Y\in C^{r\times(n-r)}$ can be arbitrary matrices. Then the following statements are equivalent:

i) $A_{X,Y}\geq 0$;

ii) $A_{11}\geq 0$，and there are $K\in C^{r\times(n-r)}$, $Y_0\in C_0^{(n-r)\times(n-r)}$ such that

$$X=\frac{1}{2}K^*(A_{11}+A_{11}^*)K+Y_0,$$

$$Y=(A_{11}+A_{11}^*)K-A_{21}^*.$$

In this case $A_{X,Y}$ has the following general form

$$A_{X,Y}=\begin{bmatrix} A_{11} & (A_{11}+A_{11}^*)K-A_{21}^* \\ A_{21} & \frac{1}{2}K^*(A_{11}+A_{11}^*)K+Y_0 \end{bmatrix},$$

Where $K\in C^{r\times(n-r)}$, $Y_0\in C_0^{(n-r)\times(n-r)}$ are arbitrary;

iii) $A_{11}\geq 0$ and $X\geq 0$，there is a contraction $\omega\in W^{r\times(n-r)}$ such that

$$Y=(A_{11}+A_{11}^*)^{\frac{1}{2}}\omega(X+X^*)^{\frac{1}{2}}-A_{21}^*.$$

In this case，$A_{X,Y}$ has the general form

$$A_{X,Y}=\begin{bmatrix} A_{11} & (A_{11}+A_{11}^*)^{\frac{1}{2}}\omega(X+X^*)^{\frac{1}{2}}-A_{21}^* \\ A_{21} & X \end{bmatrix},$$

where $X\in C_0^{(n-r)\times(n-r)}$，$\omega\in W^{r\times(n-r)}$ are arbitrary.

Proof Note that $A_{X,Y}\in C_0^{n\times n}\Leftrightarrow A_{X,Y}+A_{X,Y}^*\in HC_0^{n\times n}$,

i.e.，$\begin{bmatrix} A_{11}+A_{11}^* & A_{21}^*+Y \\ A_{21}+Y^* & X+X^* \end{bmatrix}\in HC_0^{n\times n}$.

"i)⇒ii)" It is follow from $A_{X,Y}\in HC_0^{n\times n}$ that $A_{X,Y}+A_{X,Y}^*\in HC_0^{n\times n}$. By Lemma 3，we obtain $A_{11}+A^*\in HC_0^{r\times r}$, and there exists K such that $A_{21}^*+Y=(A_{11}+A_{11}^*)K$，i.e，$Y=(A_{11}+A_{11}^*)K-A_{21}^*$，and $X+X^*-K^*(A_{11}+A_{11}^*)K\in HC_0^{(n-r)\times(n-r)}$.

Put $Y_0=X-\frac{1}{2}K^*(A_{11}+A_{11}^*)K$，we have $Y_0+Y_0^*=X+X^*-K^*(A_{11}+A_{11}^*)K\in HC_0^{(n-r)\times(n-r)}$. So $Y_0\in C_0^{(n-r)\times(n-r)}$，and $X=\frac{1}{2}K^*(A_{11}+A_{11}^*)K+Y_0$.

"ii)⇒i)" It is evident by applying Lemma 3 to the Hermite matrix $A_{X,Y}+A_{X,Y}^*$.

"i)⇔ iii)" It is not difficult by applying Lemma 1 to the Hermite matrix $A_{X,Y}+A_{X,Y}^*$.

Assume that A，B in equation (1) have the generalized singular value decompositions (GSVD):

$$A=M\sum\nolimits_A U^*,\quad B=M\sum\nolimits_B V^*, \tag{2}$$

where M is $m\times m$ nonsingular matrix，$U\in UC^{n\times n}$, $V\in UC^{p\times p}$,

$$\sum\nolimits_A=\begin{bmatrix} I_A & & \\ & S_A & \\ & & O_A \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} r \\ s \\ k-r-s \\ m-k \end{matrix},\quad \begin{matrix} r & s & n-r-s \end{matrix}$$

$$\sum\nolimits_B=\begin{bmatrix} O_B & & \\ & S_B & \\ & & I_B \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} r \\ s \\ k-r-s \\ m-k \end{matrix},\quad \begin{matrix} p-r-k & s & k-r-s \end{matrix}$$

$k=rank(A, B)$，$r=k-rankB$，$s=rankA+rankB-k$; I_A and I_B are unitmatrices; O_A, O_B are zero matrices; $S_A=diag(\alpha_1, \alpha_2, \cdots, \alpha_s)$, $S_B=diag(\beta_1, \beta_2, \cdots, \beta_s)$, $\forall\alpha_i>0, \forall\beta_i>0$.

Then equation (1) is equivalent to the system of equations

$$\begin{cases} \sum_A^* M^*XM\sum_A=U^*CU; \\ \sum_B^* M^*XM\sum_B=V^*DV. \end{cases} \tag{3}$$

We introduce the following notations and consider the following equations:

$U=(U_1,\ U_2,\ U_3)$，$V=(V_1,\ V_2,\ V_3)$

$\quad r\quad s\quad n-r-s \qquad p-r-k\quad s\quad k-r-s$

$$M^*XM=\begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}\begin{matrix} r \\ s \\ k-r-s \\ m-k \end{matrix}, \quad \begin{matrix} r & s & k-r-s & m-k \end{matrix} \tag{4}$$

$$U^*CU=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}\begin{matrix}r\\ s\\ n-r-s\end{matrix},$$

$$\quad r \quad s \quad n-r-s$$

(where $C_{ij}=U_i^*CU_j$). (5)

$$V^{\mathrm{T}}DV=\begin{bmatrix}D_{11} & D_{12} & D_{13}\\ D_{21} & D_{22} & D_{23}\\ D_{31} & D_{32} & D_{33}\end{bmatrix}\begin{matrix}p+r-k\\ s\\ k-r-s\end{matrix},$$

$$p-r-k \quad s \quad k-r-s$$

(where $D_{ij}=V_i^*DV_j$). (6)

$$H=\begin{bmatrix}U_1^*CU_1 & U_1^*CU_2S_A^{-1} & X_{13}\\ S_A^{-1}U_2^*CU_1 & S_B^{-1}V_2^*DV_2S_B^{-1} & S_B^{-1}V_2^*DV_3\\ X_{31} & V_3^*DV_2S_B^{-1} & V_3^*DV_3\end{bmatrix}=\begin{bmatrix}H_{11} & H_{12} & X_{13}\\ H_{21} & H_{22} & H_{23}\\ X_{31} & H_{32} & H_{33}\end{bmatrix}. \tag{7}$$

The following theorem is a main result in this paper.

Theorem 1 Let the generatized singular value decompositions of the matrices A and B be equation (2). Then equation (1) has a complex positive semidefinite solution if and only if

$$CU_3=0,\ U_3^*C=0,\ DV_1=0,\ V_1^*D=0, \tag{8}$$

$$S_A^{-1}U_2^*CU_2S_A^{-1}=S_B^{-1}V_2^*DV_2S_B^{-1}, \tag{9}$$

and

$$[U_1,U_2]^*C[U_1,U_2]\geq 0,\ [V_2,V_3]^*D[V_2,V_3]\geq 0. \tag{10}$$

When equation (1) has Hermite part positive semidefinite solution, the general expression of such solution is the following

$$X=(M^{-1})^*\begin{bmatrix}H & \bar{X}_{12}\\ X_{21} & X_{22}\end{bmatrix}M^{-1}, \tag{11}$$

where $H\geq 0$, and $\bar{X}_{21}\in C^{(m-k)\times k}$, $\bar{X}_{12}\in C^{k\times(m-k)}$, $\bar{X}_{22}=X_{44}\in C^{(m-k)\times(m-k)}$ such that $X\geq 0$.

Proof Since equation (1) and (3) are equivalent, so substituting equation (4),(5) and (6)into (3), then we have

$$\begin{cases}\begin{bmatrix}X_{11} & X_{12}S_A & 0\\ S_AX_{21} & S_AX_{22}S_A & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix},\\ \begin{bmatrix}0 & 0 & 0\\ 0 & S_BX_{22}S_B & S_BX_{23}\\ 0 & X_{32}S_B & X_{33}\end{bmatrix}=\begin{bmatrix}D_{11} & D_{12} & D_{13}\\ D_{21} & D_{22} & D_{23}\\ D_{31} & D_{32} & D_{33}\end{bmatrix}.\end{cases} \tag{12}$$

"⇒" If equation (1) has a solution $X\in C_0^{m\times m}$, then $M^{\mathrm{T}}XM\in C_0^{m\times m}$ satisfies equation (12). Hence

$$C_{13}=0,\ C_{23}=0,\ C_{33}=0,\ C_{31}=0,\ C_{32}=0,$$
$$D_{11}=0,\ D_{12}=0,\ D_{13}=0,\ D_{21}=0,\ D_{31}=0,$$
$$S_A^{-1}C_{22}S_A^{-1}=S_B^{-1}D_{22}S_B^{-1},$$

that is

$$CU_3=0,\ U_3^*C=0,\ V_1^*D=0,\ DV_1=0,$$
$$S_A^{-1}C_{22}S_A^{-1}=S_B^{-1}D_{22}S_B^{-1}.$$

So equation (8) and (9) are hold.

Next, noting equation (4),(12) and $C_{ij}=U_i^*CU_j$, $D_{ij}=V_i^*DV_j$, we have:

$$M^*XM=\begin{bmatrix}U_1^*CU_1 & U_1^*CU_2S_A^{-1} & X_{13} & X_{14}\\ S_A^{-1}U_2^*CU_1 & S_B^{-1}V_2^*DV_2S_B^{-1} & S_B^{-1}V_2^*DV_3 & X_{24}\\ X_{31} & V_3^*DV_2S_B^{-1} & V_3^*DV_3 & X_{34}\\ X_{41} & X_{42} & X_{43} & X_{44}\end{bmatrix}=\begin{bmatrix}H & \bar{X}_{12}\\ \bar{X}_{21} & \bar{X}_{22}\end{bmatrix}. \tag{13}$$

By $M^*XM\geq 0$ and Lemma 4, we have $H\geq 0$. On the other hand, it follows from Lemma 2 that

$$\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\geq 0,\quad \begin{bmatrix}H_{22} & H_{23}\\ H_{32} & H_{33}\end{bmatrix}\geq 0,$$

that is

$$\begin{bmatrix}I & 0\\ 0 & S_A^{-1}\end{bmatrix}[U_1,U_2]^*C[U_1,U_2]\begin{bmatrix}I & 0\\ 0 & S_A^{-1}\end{bmatrix}\geq 0,$$

$$\begin{bmatrix}S_B^{-1} & 0\\ 0 & I\end{bmatrix}[V_2,V_3]^*D[V_2,V_3]\begin{bmatrix}S_B^{-1} & 0\\ 0 & I\end{bmatrix}\geq 0,$$

i. e.,

$$[U_1,U_2]^*C[U_1,U_2]\geq 0,\ [V_2,V_3]^*D[V_2,V_3]\geq 0.$$

So equation (10) holds.

"⇐" If equation (8), (9) and (10) hold, then by equation (4) and (12), the equation (1) in $C_0^{m\times m}$ has a solution $X=(M^{-1})^*\begin{bmatrix}H & X_{12}\\ X_{21} & X_{22}\end{bmatrix}M^{-1}$, where $H\geq 0$ as equation (7), $\bar{X}_{12}\in C^{k\times(m-k)}$, $\bar{X}_{21}\in C^{(m-k)\times k}$ and $\bar{X}_{22}=X_{44}\in C^{(m-k)\times(m-k)}$, such that $X\geqslant 0$.

Remark If $C\in C_0^{n\times n}$, $D\in C_0^{p\times p}$, then equation (1) has a solution $X\in C_0^{m\times m}$ if and only if equation (8), (9) holds.

Now we give four kinds of expression of Hermite part positive semidefinite solutions for equation (1).

Theorem 2 When equation (1) has a complex positive semidefinite solution, the general Hermite part positive semidefinite solution have the following different expressions.

a) $X=(M^{-1})^*\begin{bmatrix} H & (H+H^*)K-X_{21}^* \\ \bar{X}_{21} & \frac{1}{2}K^*(H+H^*)K+Y_0 \end{bmatrix}M^{-1}$,

where $H=H(X_{13}, X_{31})$ as in equation (7), in which $X_{31}\in C^{(k-r-s)\times r}$ is arbitrary.

$$X_{13}=B_{12}B_{22}B_{23}+(B_{11}-B_{12}B_{22}B_{12}^*)^{\frac{1}{2}}\omega(B_{33}-B_{23}^*B_{22}^-B_{23})^{\frac{1}{2}}-X_{31}^*,\ B_{ij}=H_{ij}+H_{ji}^*,$$

$\bar{X}_{21}\in C^{(m-k)\times k}$, $K\in C^{k\times(m-k)}$ and $Y_0\in C_0^{(m-k)\times(m-k)}$ are arbitrary.

b) $X=(M^{-1})^*\begin{bmatrix} H & \bar{X}_{12} \\ K^*(H+H^*)-\bar{X}_{12}^* & \frac{1}{2}K^*(H+H^*)K+Y_0 \end{bmatrix}M^{-1}$,

where H, X_{31}, X_{13} are as a). $\bar{X}_{12}\in C^{k\times(m-k)}$, $K\in C^{k\times(m-k)}$ and $Y_0\in C_0^{(m-k)\times(m-k)}$ are arbitrary.

c) $X=(M^{-1})^*\begin{bmatrix} H & (H+H^*)^{\frac{1}{2}}\omega(X_{22}+\bar{X}_{22}^*)^{\frac{1}{2}}-\bar{X}_{21}^* \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix}M^{-1}$,

where H, X_{31} and X_{13} are as above, $\bar{X}_{21}\in C^{(m-k)\times k}$, $\omega\in W^{k\times(m-k)}$, $\bar{X}_{22}\in C_0^{(m-k)\times(m-k)}$ are arbitrary.

d) $X=(M^{-1})^*\begin{bmatrix} H & X_{12} \\ (\bar{X}_{22}+\bar{X}_{22}^*)^{\frac{1}{2}}\omega(H+H^*)^{\frac{1}{2}}-X_{12}^* & \bar{X}_{22} \end{bmatrix}M^{-1}$,

where H, X_{31} and X_{13} are as a). $\bar{X}_{12}\in C^{k\times(m-k)}$, $\omega\in W^{(m-k)\times k}$, $\bar{X}_{22}\in C_0^{(m-k)\times(m-k)}$ are arbitrary.

Proof By Lemma 2 and Lemma 4, these expressions are evident. So we omit the proof.

Remark

1) In $H=H(X_{13}, X_{31})$, letting $X_{13}\in C^{r\times(k-r-s)}$ be arbitrary and $X_{31}=B_{32}B_{22}^{\dagger}B_{21}+(B_{33}-B_{23}^{\dagger}B_{22}B_{23})^{\frac{1}{2}}\omega(B_{11}-B_{12}B_{22}^{\dagger}B_{12}^{\dagger})^{\frac{1}{2}}-X_{13}^*$, where $B_{ij}=H_{ij}+H_{ji}^*$, we also obtain the expression a), b), c), d) as theorem 2.

2) When $B=0$ and $D=0$, theorem 2 means that we have given four kinds of general forms of Hermite part positive semidefinite solutions for the equation $A^*XA=C$.

3 The Hermite positive semidefinite solution of the matrix equation $(A^*XA, B^*XB)=(C, D)$

This part is the particular situation of the second part. We obtain an equivalent condition and expressions of the Hermite positive semidefinite solution of the matrix equation $(A^*XA, B^*XB)=(C, D)$.

The following Lemma 5 and Lemma 6 are from Lemma 2 and Lemma 4.

Lemma 5 Let

$$A_{X_{13}}=\begin{bmatrix} A_{11} & A_{12} & X_{13} \\ A_{12}^* & A_{22} & A_{23} \\ X_{13}^* & A_{23}^* & A_{33} \end{bmatrix}\in HC^{n\times n},$$

where $A_{11}\in C^{r\times r}$, $A_{22}\in C^{s\times s}$ all A_{ij} are given and X_{13} can be arbitrary. Then $A_{X_{13}}\in HC_0^{n\times n}$ if and only if

i) $\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}\geq 0$, $\begin{bmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{bmatrix}\geq 0$;

ii) $X_{13}=A_{12}A_{22}^-A_{23}+[A_{11}-A_{12}A_{22}^{\dagger}A_{12}^*]^{\frac{1}{2}}\omega[A_{33}-A_{23}^*A_{22}^{\dagger}A_{23}]^{\frac{1}{2}}$.

where $\omega\in W^{r\times(n-r-s)}$ is arbitrary.

Lemma 6 Let $A_{X,Y}\in C^{n\times n}$ have the following form

$$A_{X,Y}=\begin{bmatrix} A_{11} & Y \\ Y^* & X \end{bmatrix},$$

where $A_{11}\in C^{r\times r}$ is given and $X\in C^{(n-r)\times(n-r)}$, $Y\in C^{r\times(n-r)}$ can be arbitrary matrices. Then following statements are equivalent:

i) $A_{X,Y}\in HC_0^{n\times n}$;

ii) $A_{11}\in HC_0^{r\times r}$ and there are $K\in C^{r\times(n-r)}$, $Y_0\in HC_0^{(n-r)\times(n-r)}$ such that $X=K^*A_{11}K+Y_0$, $Y=A_{11}K$.

In this case $A_{X,Y}$ has the following general form

$$A_{X,Y}=\begin{bmatrix} A_{11} & A_{11}K \\ K^*A_{11} & K^*A_{11}K+Y_0 \end{bmatrix},$$

where $K\in C^{r\times(n-r)}$, $Y_0\in HC_0^{(n-r)\times(n-r)}$ are arbitrary;

iii) $A_{11}\in HC_0^{r\times r}$, $X\in HC_0^{(n-r)\times(n-r)}$ and there is a contraction $\omega\in W^{r\times(n-r)}$ such that $Y=A_{11}^{\frac{1}{2}}\omega X^{\frac{1}{2}}$.

In this case $A_{X,Y}$ has the general form

$$A_{X,Y}=\begin{bmatrix} A_{11} & A_{11}^{\frac{1}{2}}\omega X^{\frac{1}{2}} \\ X^{\frac{1}{2}}\omega^* A_{11}^{\frac{1}{2}} & X \end{bmatrix},$$

where $X\in HC_0^{(n-r)\times(n-r)}$, $\omega\in W^{r\times(n-r)}$ are arbitrary.

The following results are from Theorem 1 and Theorem 2. So we also omit the proof.

Theorem 3 Let the generatized singular value decompositions of the matrices A and B be equation (2). Then equation (1) has a Hermite positive semidefinite solution if and only if

$$CU_3=0,\ U_3^*C=0,\ DV_1=0,\ V_1^*D=0; \tag{14}$$

$$S_A^{-1}U_2^*CU_2S_A^{-1}=S_B^{-1}V_2^*DV_2S_B^{-1}, \tag{15}$$

and $C\in HC_0^{n\times n}$, $D\in HC_0^{p\times p}$. (16)

When equation (1) has Hermite positive semidefinite solution, the general expressions of such solution are the following

$$X=(M^{-1})^*\begin{bmatrix} H & \bar{X}_{12} \\ \bar{X}_{12}^* & \bar{X}_{22} \end{bmatrix}M^{-1}, \tag{17}$$

where $H\in HC_0^{k\times k}$,

$$H=\begin{bmatrix} U_1^*CU_1 & U_1^*CU_2S_A^{-1} & X_{13} \\ S_A^{-1}U_2^*CU_1 & S_B^{-1}V_2^*DV_2S_B^{-1} & S_B^{-1}V_2^*DV_3 \\ X_{13}^* & V_3^*DV_2S_B^{-1} & V_3^*DV_3 \end{bmatrix}=$$

$$\begin{bmatrix} H_{11} & H_{12} & X_{13} \\ H_{12}^* & H_{22} & H_{23} \\ X_{13}^* & H_{23}^* & H_{33} \end{bmatrix}, \tag{18}$$

and $\bar{X}_{12}\in C^{k\times(m-k)}$, $\bar{X}_{22}\in C_0^{(m-k)\times(m-k)}$, such that $X\in HC_0^{m\times m}$.

Theorem 4 When equation (1) has a Hermite positive semidefinite solution, the general Hermite positive semidefinite solution has the following different expressions.

a) $$X=(M^{-1})^*\begin{bmatrix} H & HK \\ K^*H & K^*HK+Y_0 \end{bmatrix}M^{-1},$$

where $H=H(X_{13})$ as (18),

$$X_{13}=H_{12}H_{22}^+H_{23}+(H_{11}-H_{12}H_{22}^+H_{12}^*)^{\frac{1}{2}}\omega(H_{33}-H_{23}^*H_{22}^+H_{23})^{\frac{1}{2}},$$

$K\in C^{k\times(m-k)}$ and $Y_0\in HC_0^{(m-k)\times(m-k)}$ are arbitrary.

b) $$X=(M^{-1})^*\begin{bmatrix} H & H^{\frac{1}{2}}\omega\bar{X}_{22}^{\frac{1}{2}} \\ \bar{X}_{22}^{\frac{1}{2}}\omega^*H^{\frac{1}{2}} & \bar{X}_{22} \end{bmatrix}M^{-1},$$

where H and X_{13} are as above, $\omega\in W^{k\times(m-k)}$, $X_{22}\in HC_0^{(m-k)\times(m-k)}$ are arbitrary.

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我校举行湖南工业大学期刊社成立大会

2008年1月14日，湖南工业大学期刊社成立暨工作会议在我校科技大楼7楼会议室举行。校领导侯清麟、王汉青、罗定提出席会议并发表讲话，部分相关部门负责同志参加会议并对期刊社的成立表示祝贺。

党委书记侯清麟在讲话中对期刊社坚持正确的办刊方向给予了肯定，他强调，原学报编辑部有着“严谨、求实、创新”的优良传统，对繁荣学术、鼓励科研创新做出了积极贡献。他要求期刊社的同志要不断提高编辑学术修养，反复甄别稿件，努力提高刊物质量，为广大教学科研工作者营造良好的学术平台，让期刊社成为反映我校最新教学科研成果的园地。校长王汉青向期刊社提出了“把准方向、多出精品、突出特色、提升水平”的十六字要求，他希望期刊社的同志要扎扎实实开展调查研究，勇于创新，主动吸纳优质稿件，不断推出精品；要注重资源整合，杜绝人情稿，做出特色和品牌。副校长罗定提、湖南工业大学期刊社的主要负责人在庆祝大会上相继作了发言。

期刊社的成立，将为我校学科建设、人才培养和科研工作做出新的贡献，也有助于提升校园文化品质，营造浓郁的学术氛围。